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SIXTH-ORDER COMPACT DIFFERENCING WITH STAGGERED BOUNDARY SCHEMES AND 3(2) BOGACKI-SHAMPINE PAIRS FOR PRICING FREE-BOUNDARY OPTIONS

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Abstract. We propose a stable sixth-order compact finite difference scheme coupled with a fifth-order staggered boundary scheme and the Runge-Kutta adaptive time stepping based on 3(2) Bogacki-Shampine pairs for pricing American options. To compute the free-boundary simultaneously and precisely with the option value and Greeks, we introduce a logarithmic Landau transformation and then remove the convective term in the pricing model by introducing the delta sensitivity, so that an efficient sixth-order compact scheme can be easily implemented. The main challenge in coupling the sixth order compact scheme in discrete form is to efficiently account for the near-boundary scheme. In this study, we introduce novel fifth and sixth-order Dirichlet near-boundary schemes that are suitable for solving our model. The optimal exercise boundary and other boundary values are approximated using a high-order analytical approximation that is obtained from a novel fifth-order staggered boundary scheme. Furthermore, we investigate the smoothness of the first and second-order derivatives of the optimal exercise boundary which is obtained from the high-order analytical approximation. Coupled with the adaptive time integration method, the interior values are then approximated using the sixth order compact schemes. As such, the expected convergence rate is reasonably achieved, and the present numerical scheme is very fast in computation and gives highly accurate solutions with very coarse grids.

Key words. Sixth-order compact finite difference, 3(2) Bogacki and Shampine pairs, Dirichlet and Neumann boundary conditions, options price, Delta sensitivity, optimal exercise boundary.

1. Introduction

Under risk neutral probability, the model governing the American style put options value $P(S,t)$ and the optimal exercise boundary $s_f(t)$ can be expressed as

$$\frac{\partial P(S,t)}{\partial t} - \frac{\sigma}{2}\frac{\partial^2 P(S,t)}{\partial S^2} - r\frac{\partial P(S,t)}{\partial S} + rP(S,t) = 0, \quad S > s_f(t), \quad t > 0; \tag{1}$$

$$P(S,t) = E - S, \qquad S < s_f(t); \tag{2}$$

$$P(s_f(t),t) = E - s_f(t), \qquad \frac{\partial P(s_f(t),t)}{\partial S} = -1; \tag{3}$$

$$P(\infty,t) = 0, \qquad \frac{\partial P(\infty,t)}{\partial S} = 0; \tag{4}$$

$$s_f(0) = E, \qquad P(S,0) = \max(E-S,0). \tag{5}$$

Here, S is the asset price, T is the time to maturity E is the strike price, σ is the volatility, and r is the interest rate. It can be seen that the above model is a free boundary problem since $s_f(t)$ varies with time. There is no closed-form solution for this model and therefore, it must be solved using numerical or semi-analytical methods. The solution framework for the above model can be formulated as a linear complementary problem, with a penalty method or front-fixing approach. However,

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in a linear complementary framework also known as variational inequality and the penalty method, constraints are imposed and it has been shown that the obtained optimal exercise boundary is not very accurate. In the front-fixing approach [38], one may apply the Landau transformation

$$x = \ln \frac{S}{s_f(t)}, \qquad P(e^x s_f(t), t) = U(x,t), \tag{6}$$

to (6) and obtain a fixed free-boundary equation as follows:

$$\frac{\partial U(x,t)}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 U(x,t)}{\partial x^2} - \xi_t \frac{\partial U(x,t)}{\partial x} + rU(x,t) = 0, \qquad x > 0; \tag{7}$$

where

$$\xi_t = r - \frac{s_f'(t)}{s_f(t)} - \frac{\sigma^2}{2}. \tag{8}$$

Although the free-boundary is now fixed at $x = 0$, it is worth observing that (7) becomes a nonlinear partial differential equation with a singular coefficient. This is because the derivative of the optimal exercise boundary involved in the coefficient of the convective term is not continuous at payoff. This irregularity presents a source of non-smoothness in the model and the convective term could further introduce substantial errors when using numerical approximation. To remove $\frac{\partial U(x,t)}{\partial x}$, we in this study introduce the delta sensitivity $W(x,t)$ as

$$W(x,t) = \frac{\partial U(x,t)}{\partial x}. \tag{9}$$

Thus, we obtain a system of two fixed-free boundary partial differential equations (PDEs) for the option value and delta sensitivity that is suitable for implementing an efficient sixth-order compact scheme as follows:

$$\frac{\partial U(x,t)}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 U(x,t)}{\partial x^2} - \xi_t W(x,t) + rU(x,t) = 0, \qquad x > 0; \tag{10}$$

$$\frac{\partial W(x,t)}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 W(x,t)}{\partial x^2} - \xi_t \frac{\partial^2 U(x,t)}{\partial x^2} + rW(x,t) = 0, \qquad x > 0; \tag{11}$$

$$U(x,t) = E - e^x s_f(t), \qquad W(x,t) = -e^x s_f(t), \qquad x \le 0; \tag{12}$$

with initial and boundary conditions:

$$U(0,t) = E - s_f(t), \qquad W(0,t) = -s_f(t); \tag{13}$$

$$U(\infty,t) \cong 0, \qquad W(\infty,t) \cong 0; \tag{14}$$

$$U(x,0) = 0, \qquad W(x,0) = 0, \qquad x > 0; \tag{15}$$

$$U(0,0) = 0, \qquad \lim_{x \to 0^+} W(x,0) = 0. \tag{16}$$

Under an assumption of sufficient smoothness, a high-order numerical scheme can be used to obtain a more accurate numerical solution with very coarse grids. This feature could be beneficial in saving computational time and improving complexity in high dimensional context. However, the non-smoothness in the transformed model hampers this possibility [1, 6, 22, 33]. Several authors have tried to implement high-order numerical scheme for solving American options using the front-fixing approach. Hajipour and Malek [15] implemented an efficient fifth-order WENO-BDF3 scheme for solving the American options but only recovered a second-order accurate solution. Tangman et al. [33] implemented a fourth-order numerical scheme