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# AN EFFICIENT ROTATIONAL PRESSURE-CORRECTION SCHEME FOR THE 2D/3D NAVIER-STOKES/DARCY COUPLING PROBLEM

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**Abstract.** This article proposes and analyzes a rotational pressure-correction method for the Navier-Stokes/Darcy(NSD) system with Beavers-Joseph-Saffman-Jones interface conditions. This method mainly solves the Navier-Stokes/Darcy problem in two steps. The first step is the viscous step. The intermediate velocity can be obtained after the pressure gradient is explicitly processed by the algorithm. The second step is the projection step, which first projects the intermediate velocity onto a divergence-free space, and then corrects the velocity and pressure. The main advantage of these methods is that they have first/second order accuracy and do not have the incompressibility constraint of NSD system. For solving the Navier-Stokes equations, each time step requires only one vector-valued elliptic equation and one scalar-valued Poisson equation. Therefore, this method has high computational efficiency. Compared with other traditional related methods, this method is no longer affected by any artificial boundary conditions, and can achieve the optimal convergence order. Finally, unconditional stability and long time stability are established. 2D/3D numerical experiments are presented to illustrate the features of the proposed method and verify the results of the theoretical analysis.

**Key words.** First-order/second-order temporal scheme, rotational pressure-correction scheme, Navier-Stokes/Darcy system, stability, 2D/3D numerical experiments.

## 1. Introduction

The NSD system is one of the classical equations of fluid mechanics, because it describes the physical phenomena of fluid motion. It may be used to simulate surface water flow, subsurface oil and groundwater flow, as well as flow in porous media, such as [1, 2, 3, 4, 5]. Due to the coupling of free flow and flow in porous media, the complex geometric shape of free flow, refined space-time scale, strong heterogeneity of physical parameters and uncertainty of experimental data, the mathematical research of this system has always been a very difficult challenge.

In most important applications, it is difficult to solve the exact solution of multi-domain, multi-physics field coupled NSD system. Therefore, the efficient numerical solution of the system is particularly important. Most methods are designed for the development of an approximate solution to the NSD system, including coupled finite element methods [2, 6, 7, 8, 9, 10, 11], domain decomposition methods [12, 13, 14, 15, 16, 17, 18, 19, 20, 21], Lagrange multiplier methods [22, 23], two-grid methods [5, 24], implicit-explicit method [25, 26, 27, 28], discontinuous Galerkin methods [29, 30, 31, 32, 33], mortar discretizations [34, 35, 36], boundary integral methods [37, 38], and others [39, 40, 41, 42, 43, 44, 45]. These numerical methods have been widely devoted to achieve their required accuracy in certain practice. While they have proven to be very successful, a theory to ensure their long time stability is still being developed. In recent years, some efficient second-order (in time) accurate methods have been developed and investigated for a NSD system [42, 46]. These methods establish an unconditional and uniform stability and further

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lead to a uniform control of errors, which is highly desirable for modeling many physical processes. In particular, two second-order time method for coupled NSD systems and optimal theoretical results are studied in [46]. These results provide a theoretical support for second-order large time step method for the NSD system.

It is well known that a lot of difficulties are arising from the coupled system: the mixed nonlinear problem, their shared rigid interface, the energy dissipation derived from a decoupling strategy, and their complex system existing in itself. Thus, it is not easy to solve efficiently the coupled system with the Navier-Stokes equations because of the saddle point structure induced by the incompressibility constraint [47, 48, 49, 50, 51, 52]. For the past few years, a projection method has attracted more and more attention from researchers because of its simplicity and efficiency[53, 54, 55, 56]. The projection algorithm[57] has undergone some evolution and has been well further developed. Firstly, Chorin-Temam's algorithm was improved by making the pressure explicit in the viscous step and by correcting it in the projection step. Hereafter, the same fundamental idea of decomposing vector fields into a divergence-free part and a gradient has remained effective for solving the Navier-Stokes equations [58, 59]. For incompressible flow, it is effective to decouple the system of pressure from that of velocity by using the projection method [60]. These methods usually consist of two substeps. One substep explicitly satisfies the Laplacian expression of the velocity or pressure gradient, and the other substep implicitly corrected . In [61], the author gives a second order in time incremental pressure correction finite element method for the Navier-Stokes/Darcy problem. In this method, the Navier-Stokes/Darcy problem is solved in three steps: a convection-diffusion step, a projection correction (incremental pressure correction) step and a Darcy step. In [62], a first order linearized pressure correction projection method is proposed and analyzed for the time-dependent diffusive Peterlin viscoelastic model, which can describe the unsteady behavior of some incompressible polymeric fluids in two dimensions. Details on the various projection method can be found at [63, 64, 65, 66].

Here, we restrict ourselves to the rotational pressure-correction method for the NSD system. The most challenging issue is how to develop the proper rotational pressure-correction method. One of the main difficulties in decoupling the coupled system related to the incompressible flow is that this system has a complicated boundary condition. Most of these schemes imply an artificial condition not satisfied by the exact pressure, which induces a numerical boundary layer, and, in turn, results in a loss of accuracy. In addition to the complexity of the coupling problem, the analysis of nonlinear terms for the incompressible Navier-Stokes equation is another difficulty, and we need to pay attention to the analysis skills to overcome it. The projection method has been widely used because of its efficiency and simplicity. However, the rigorous error analysis of coupled systems with incompressible Navier-Stokes equations still needs further study.

In this article, based on the key idea of the rotational pressure-correction method for the Stokes problem with an open boundary condition in [66, 67], we propose and rigorously analyze this scheme to solve the coupled NSD system. A first-order backward Euler and second-order backward difference formulas are utilized to discretize the time derivative while the finite elements are used to treat the spatial discretization. The central advantage of our approach is a time-dependent version of domain decomposition, and has first-order/second-order accuracy without the incompressibility constraint in the Navier-Stokes system. Moreover, the negative