INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 21, Number 6, Pages 879–909

doi: 10.4208/ijnam2024-1035

VARIABLE TIME STEP METHOD OF DAHLQUIST, LINIGER AND NEVANLINNA (DLN) FOR A CORRECTED SMAGORINSKY MODEL

FARJANA SIDDIQUA AND WENLONG PEI*

Abstract. Turbulent flows strain resources, both memory and CPU speed. A family of second-order, G-stable time-stepping methods proposed by Dahlquist, Liniger, and Nevanlinna (the DLN method) has great accuracy and allows large time steps, requiring less memory and fewer FLOPS. The DLN method can also be implemented adaptively. The classical Smagorinsky model, as an effective way to approximate a resolved mean velocity, has recently been corrected to represent a flow of energy from unresolved fluctuations to the resolved mean velocity. In this paper, we apply the DLN method to one corrected Smagorinsky model and provide a detailed numerical analysis of the stability and consistency. We prove that the numerical solutions under arbitrary time step sequences are unconditionally stable in the long term and converge in second order. We also provide error estimates under certain time-step conditions. Numerical tests are given to confirm the rate of convergence and also to show that the adaptive DLN algorithm helps to control numerical dissipation so that a flow of energy from unresolved fluctuations to the resolved mean velocity is visible.

Key words. Eddy viscosity, corrected Smagorinsky model, complex turbulence, backscatter, the DLN method, G-stability, variable time-stepping.

1. Introduction

Herein we give an analysis of the method of Dahlquist, Liniger, and Nevanlinna [19] (the DLN method) for the corrected Smagorinsky model (CSM henceforth) [59] with variable time steps. Time adaptivity (adjusting time steps based on certain criteria) is an effective way to balance accuracy and time efficiency.

Eddy viscosity (EV) models are the most common approaches to depict the average turbulent flow of Navier-Stokes equations (NSE). Various eddy viscosity models in practical settings are proposed for analytical and numerical study [4,21,22,28,29]. In large eddy simulation (LES), backscatter is the study and measurement of the energy transfer process from small, unresolved turbulent scales to large, resolved scales in a computational fluid dynamics (CFD) simulation. Unfortunately, most EV models have difficulties in simulating backscatter or complex turbulent flow not at statistical equilibrium due to the neglect of the intermittent energy flow from fluctuations back to means. To overcome this defect, Jiang and Layton [33] derive a new eddy viscosity model from an equation describing the evolution of variance in a turbulent flow. Rong, Layton, and Zhao [57] extended the usual Baldwin-Lomax model so that the new model can account for statistical backscatter[1] without artificial negative viscosities. Recently, Siddiqua and Xie [59] have corrected the classical Smagorinsky model [60] with no new fitting parameters to reflect a flow of energy from unresolved fluctuations to means in the CSM. Most recently, Dai, Liu, Liu, Jiang, and Chen [18] proposed a new dynamic Smagorinsky model by an artificial

Received by the editors on February 23, 2024 and, accepted on July 22, 2024.
2000 *Mathematics Subject Classification.* 65M12, 65M22, 65M60, 76M10.
*Corresponding author.
[1]We will refer to the movement of energy from fluctuations back to means statistical backscatter when using ensemble averaging.

neural network for the prediction of outdoor airflow and pollutant dispersion. In the report, we give a detailed numerical analysis of the CSM [59] under arbitrary non-uniform time grids. Given bounded flow domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, time interval $[0, T]$, and the prescribed body force $f(x, t)$, the pair $(w(x, t), q(x, t))$ approximate an ensemble average pair of velocity and pressure of Navier-Stokes solutions $(\overline{u}, \overline{p})$ and is governed by the following system

(1)
$$\begin{cases} w_t - \frac{C_s^4 \delta^2}{\mu^2} \Delta w_t + w \cdot \nabla w - \nu \Delta w - \nabla \cdot \big((C_s \delta)^2 |\nabla w| \nabla w \big) = f, & (x,t) \in \Omega \times (0, T] \\ \nabla \cdot w = 0, & (x,t) \in \Omega \times (0, T] \\ w(x, 0) = w_0(x), & x \in \Omega \\ w(x, t) = 0, & (x,t) \in \partial\Omega \times (0, T] \\ \int_\Omega p(x,t) dx = 0, & t \in (0, T] \end{cases}.$$

This is an eddy viscosity model. Constant $C_s \approx 0.1$ is suggested by Lilly [42]. δ is a length scale (or grid-scale) and μ is a constant from Kolmogorov-Prandtl relation [38, 54]. ν is the kinematic viscosity and $\nu_T = (C_s \delta)^2 |\nabla w|$ is the turbulent viscosity. $|\cdot|$ is the Euclidian norm on $\mathbb{R}^d$. The viscous term $\nabla \cdot \big((C_s \delta)^2 |\nabla w| \nabla w \big)$ in (1) comes from the classic Smagorinsky model and the kinetic energy penalization $-\frac{C_s^4 \delta^2}{\mu^2} \Delta w_t$ in (1) is newly added for the CSM. All other terms in (1) are from standard Navier-Stokes equations (NSE). In [59], the CSM model derivation and some basic properties of the CSM are developed, and two algorithms for its numerical simulation are proposed. However, the significant backscatter of model dissipation is not observed in specific examples except for Linearized Crank-Nicolson time discretization [59, page 21-22]. Besides that, constant time discretization in their algorithms (Linearized Crank-Nicolson time-stepping scheme) excludes the use of time adaptivity since the solution pattern (in terms of stability and convergence) under extreme time step ratios is hard to expect[1]. Dahlquist, Liniger, and Nevanlinna designed a one-parameter family of one-leg, second-order methods for evolutionary equations [19]. This family of one-leg methods (For convenience, we call this family the DLN method.) is proved to be G-stable (non-linear stable) under any arbitrary time grids [14–16] and hence ideal choice for time discretization of fluid models[2]. Herein we apply the fully discrete DLN algorithm (finite element space discretization) for the CSM in (1) and present a complete numerical analysis of the algorithm in Section 4. We prove that the numerical solutions on arbitrary time grids are unconditionally long-term stable, and converge to exact solutions at second order with moderate time step restrictions. Let $\{t_n\}_{n=0}^N$ be the time grids on interval $[0, T]$ and $k_n = t_{n+1} - t_n$ the local time step. w_n^h and q_n^h are numerical approximations of velocity and pressure at time t_n of the CSM in (1) respectively on certain finite element space with the diameter h. The fully discrete DLN algorithm

[1]In [19], the linearized Crank-Nicolson scheme and applying to the problem $y'(t) = \lambda(t) y(t)$ with $\mathrm{Re}(\lambda(\mathrm{t})) < 0$ and $\lambda(t_{2n}) = 0$. Under certain time step sequence ($k_n = 7$ and $k_{2n+1} = 1/2$), the sequence of numerical solutions satisfy $y_{2n} = (-2)^n y_0$, which implies the scheme is not stable.

[2]To the best of our knowledge, the DLN method is the *only* variable multi-step method which is both non-linearly stable and second-order accurate.