INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 10, Number 1, Pages 154–177

A PRIORI ERROR ESTIMATES FOR SEMI-DISCRETE DISCONTINUOUS GALERKIN METHODS SOLVING NONLINEAR HAMILTON-JACOBI EQUATIONS WITH SMOOTH SOLUTIONS

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Abstract. In this paper, we provide a priori L^2 error estimates for the semi-discrete discontinuous Galerkin method [3] and the local discontinuous Galerkin method [22] for one- and two-dimensional nonlinear Hamilton-Jacobi equations with smooth solutions. With a special Gauss-Radau projection, the optimal error estimates on rectangular meshes are obtained.

Key words. Hamilton-Jacobi equations, discontinuous Galerkin method, local discontinuous Galerkin method, a priori error estimates.

1. Introduction

In this paper, we are interested in the a priori L^2 error estimates of the semi-discrete discontinuous Galerkin (DG) and local discontinuous Galerkin (LDG) methods for smooth solutions of nonlinear Hamilton-Jacobi (HJ) equations in the one-dimensional case

$$\phi_t + H(\phi_x, x) = 0, \qquad \phi(x,0) = \phi^0(x) \tag{1}$$

and in the two-dimensional case:

$$\phi_t + H(\phi_x, \phi_y, x, y) = 0, \qquad \phi(x,y,0) = \phi^0(x,y). \tag{2}$$

The Hamiltonian H is assumed to be a smooth function of all the arguments. When there is no ambiguity, we also take the concise notation $H(\phi_x) = H(\phi_x, x)$ and $H(\phi_x, \phi_y) = H(\phi_x, \phi_y, x, y)$.

The DG method is a class of finite element methods using completely discontinuous piecewise polynomial space for the numerical solution in the spatial variables. It can be discretized in time by the explicit and nonlinearly stable high order Runge-Kutta time discretization [20], resulting in the so-called RKDG method. The RKDG method was first developed for nonlinear hyperbolic conservation laws by Cockburn et al. in [8, 7, 5, 9]. Later it was generalized to the LDG method for solving convection-diffusion equations by Cockburn and Shu [10].

The time-dependent Hamilton-Jacobi (HJ) equations (1) and (2) are closely related to the conservation laws. In the one-dimensional case, they are equivalent if one takes the spatial derivative in (1) and writes out the equation satisfied by $u = \phi_x$. It is thus not surprisingly that many successful numerical methods for the conservation laws have been adapted to solve the Hamilton-Jacobi equations. For finite difference schemes, the high order essentially non-oscillatory (ENO) and weighted ENO (WENO) schemes [18, 14, 25] are such examples. However, it is less straightforward to adapt DG schemes to solve the Hamilton-Jacobi equations, since

Received by the editors December 30, 2011.

2000 *Mathematics Subject Classification.* 65N30.

Research partially supported by DOE grant DE-FG02-08ER25863, NSF grant DMS-1112700 and NSFC grants 11071234 and 91130016.

the nonlinear Hamiltonian H prevents a direct integration by parts. Hu and Shu developed a DG scheme [13] for solving the nonlinear Hamilton-Jacobi equations, which is based on the Runge-Kutta discontinuous Galerkin (RKDG) method for solving conservation laws. They first solve the conservation law equation satisfied by $u = \phi_x$ with the standard DG method, which can determine ϕ for each element up to a constant, and then the missing constant is obtained by integration either in time or from the boundary. In two dimensions, this scheme involves a least square procedure to obtain ϕ from the numerical approximations of $u = \phi_x$ and $v = \phi_y$, as they may not satisfy the compatibility condition $u_y = v_x = \phi_{xy}$. Later, Li and Shu [16] reinterpreted the method in [13] by using a curl-free subspace for the discontinuous Galerkin method in the two-dimensional case to avoid the least squares procedure. The two algorithms in [13] and [16] are mathematically equivalent, however the latter avoids the least square procedure and also uses a smaller finite element space, resulting in a significant simplification in implementation with a reduced cost. The DG scheme in [13] achieves the optimal k-th order of accuracy for $u = \phi_x$ (and also $v = \phi_y$ in two dimensions), however the optimal $(k+1)$-th order accuracy for ϕ is not always observed numerically when k-th degree piecewise polynomial space is used. For the one-dimensional case, the error estimates for conservation laws in [23, 21, 24] can be directly applied, yielding k-th order error accuracy for the upwind fluxes and $(k - \frac{1}{2})$-th order error accuracy for general numerical fluxes for the derivative $u = \phi_x$ when k-th degree piecewise polynomial space is used. For the two-dimensional case, we can follow the a priori error estimates for $u = \phi_x$ and $v = \phi_y$ in the DG curl-free subspace, however only $(k - \frac{1}{2})$-th order accuracy can be obtained either for the upwind fluxes or for general fluxes, since the special projections need for the optimal error estimates in two dimensions cannot be defined in the curl-free subspace.

More recently, Cheng and Shu in [3] proposed a DG method for directly solving Hamilton-Jacobi equations without going through the derivatives $u = \phi_x$ and $v = \phi_y$. Also, Yan and Osher [22] designed a direct LDG method for solving Hamilton-Jacobi equations. Numerically, optimal order error accuracy has been observed for both of these two methods. For linear Hamiltonians, the DG and LDG methods in [3] and [22] are equivalent to those for solving conservation laws, hence stability and error estimates can be obtained following the techniques for conservation laws. However, for nonlinear Hamiltonians, the methods in [3] and [22] are distinct from the DG methods for conservation laws. In this paper, we follow and generalize the techniques in [23, 21, 24] to obtain a priori L^2 error estimates for the DG and LDG methods in [3] and [22] for directly solving nonlinear Hamilton-Jacobi equations with smooth solutions.

The paper is organized as follows. In Section 2, we introduce notations, definitions and auxiliary results used later in this paper. In Section 3, we obtain a priori error estimates for the one-dimensional Hamilton-Jacobi equations. In Section 4, we follow the same line as the one-dimensional case to obtain a priori error estimates for the two-dimensional Hamilton-Jacobi equations. Concluding remarks are given in Section 5.

2. Notations, definitions and auxiliary results

In this section, we follow [21, 24] to first introduce notations and definitions to be used later in this paper and also present some auxiliary results. We use a special Gauss-Radau projection as in [24], and present certain interpolation and inverse properties for the finite element spaces that will be used in the error analysis.