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SECOND ORDER PARAMETER-UNIFORM CONVERGENCE FOR A FINITE DIFFERENCE METHOD FOR A SINGULARLY PERTURBED LINEAR PARABOLIC SYSTEM

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Abstract. A singularly perturbed linear system of second order partial differential equations of parabolic reaction-diffusion type with given initial and boundary conditions is considered. The diffusion term of each equation is multiplied by a small positive parameter. These singular perturbation parameters are assumed to be distinct. The components of the solution exhibit overlapping layers. Shishkin piecewise-uniform meshes are introduced, which are used in conjunction with a classical finite difference discretisation, to construct a numerical method for solving this problem. It is proved that in the maximum norm the numerical approximations obtained with this method are first order convergent in time and essentially second order convergent in the space variable, uniformly with respect to all of the parameters.

Key Words. Singular perturbation problems, parabolic problems, boundary layers, uniform convergence, finite difference scheme, Shishkin mesh.

1. Introduction

The following parabolic initial-boundary value problem is considered for a singularly perturbed linear system of second order differential equations

$$\frac{\partial \vec{u}}{\partial t} - E\frac{\partial^2 \vec{u}}{\partial x^2} + A\vec{u} = \vec{f}, \text{ on } \Omega, \ \vec{u} \text{ given on } \Gamma, \tag{1}$$

where $\Omega = \{(x,t) : 0 < x < 1, 0 < t \le T\}$, $\overline{\Omega} = \Omega \cup \Gamma$, $\Gamma = \Gamma_L \cup \Gamma_B \cup \Gamma_R$ with $\vec{u}(0,t) = \vec{\phi}_L(t)$ on $\Gamma_L = \{(0,t) : 0 \le t \le T\}$, $\vec{u}(x,0) = \vec{\phi}_B(x)$ on $\Gamma_B = \{(x,0) : 0 \le x \le 1\}$, $\vec{u}(1,t) = \vec{\phi}_R(t)$ on $\Gamma_R = \{(1,t) : 0 \le t \le T\}$. Here, for all $(x,t) \in \overline{\Omega}$, $\vec{u}(x,t)$ and $\vec{f}(x,t)$ are column $n-$vectors, E and $A(x,t)$ are $n \times n$ matrices, $E = \text{diag}(\vec{\varepsilon})$, $\vec{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n)$ with $0 < \varepsilon_i \le 1$ for all $i = 1, \ldots, n$. The ε_i are assumed to be distinct and, for convenience, to have the ordering

$$\varepsilon_1 < \cdots < \varepsilon_n.$$

Cases with some of the parameters coincident are not considered here.
The problem (1) can also be written in the operator form

$$\vec{L}\vec{u} = \vec{f} \text{ on } \Omega, \ \vec{u} \text{ given on } \Gamma,$$

where the operator $\vec{L}$ is defined by

$$\vec{L} = I\frac{\partial}{\partial t} - E\frac{\partial^2}{\partial x^2} + A,$$

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where I is the identity matrix. The reduced problem corresponding to (1) is defined by

$$\frac{\partial \vec{u}_0}{\partial t} + A\vec{u}_0 = \vec{f}, \text{ on } \Omega, \ \ \vec{u}_0 = \vec{u} \text{ on } \{(x,0) : 0 < x < 1\}. \tag{2}$$

For a general introduction to parameter-uniform numerical methods for singular perturbation problems, see [7], [9] and [1]. The piecewise-uniform Shishkin meshes $\Omega^{M,N}$ in the present paper have the elegant property that they reduce to uniform meshes when the parameters are not small. The problem posed in the present paper is also considered in [3], where parameter uniform convergence in the maximum norm is proved, which is first order in time and essentially first order in space. The meshes used there are different from those in the present paper. The main result of the present paper is established in [6] for the special case $n = 1$ and in [5] for $n = 2$. The proof in the present paper of first order convergence in the time variable and essentially second order convergence in the space variable, for general n, draws heavily on the analogous result in [8], where a slightly weaker result is proved for a reaction-diffusion system. The final result in the present paper is that the error in the maximum norm is bounded by $C(M^{-1} + (N^{-1} \ln N)^2)$, where C is a constant which is independent of the singular perturbation parameters $\vec{\varepsilon}$ and of the mesh parameters M, N. It is the factor $\ln N$ here, which makes the convergence essentially rather than fully second order, but it has little significance in practice.

The plan of the paper is as follows. In the next three sections both standard and novel bounds on the smooth and singular components of the exact solution are obtained. The sharp estimates for the singular component in Lemma 4.3 are proved by mathematical induction, while interesting orderings of the points $x_{i,j}^{(s)}$ are established in Lemma 4.2. In Section 5 piecewise-uniform Shishkin meshes are introduced. In Section 6 the discrete problem is defined and a discrete maximum principle, discrete stability properties and a comparison principle are established. In Section 7 an expression for the local truncation error is derived and standard estimates are stated. In Section 8 parameter-uniform estimates for the local truncation error of the smooth and singular components are obtained by means of a sequence of lemmas. The section culminates with the statement and proof of the required parameter-uniform error estimate in the maximum norm.

2. Solutions of the continuous problem

Standard theoretical results on the solutions of (1) are stated, without proof, in this section. See [2] and [4] for more details. For all $(x,t) \in \overline{\Omega}$, it is assumed that the components $a_{ij}(x,t)$ of $A(x,t)$ satisfy the inequalities

$$a_{ii}(x,t) > \sum_{\substack{j \neq i \\ j=1}}^{n} |a_{ij}(x,t)| \text{ for } 1 \le i \le n, \text{ and } a_{ij}(x,t) \le 0 \text{ for } i \neq j \tag{3}$$

and, for some α,

$$0 < \alpha < \min_{\substack{(x,t) \in \overline{\Omega} \\ 1 \le i \le n}} \left(\sum_{j=1}^{n} a_{ij}(x,t) \right). \tag{4}$$

It is also assumed, without loss of generality, that

$$\max_{1 \le i \le n} \sqrt{\varepsilon_i} \le \frac{\sqrt{\alpha}}{6}. \tag{5}$$