INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 10, Number 1, Pages 1–23

EQUIVALENT A POSTERIORI ERROR ESTIMATES FOR A CONSTRAINED OPTIMAL CONTROL PROBLEM GOVERNED BY PARABOLIC EQUATIONS

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Abstract. In this paper, we study adaptive finite element approximation in the backward Euler scheme for a constrained optimal control problem by parabolic equations on multi-meshes. The control constraint is given in an integral sense: $K = \{u(t) \in L^2(\Omega) : a \leq \int_\Omega u(t) \leq b\}$. We derive equivalent a posteriori error estimates with lower and upper bounds for both the state and the control approximation, which are used as indicators in adaptive multi-meshes finite element scheme. The error estimates are then implemented and tested with promising numerical experiments.

Key Words. constrained optimal control problem, adaptive finite element approximation, equivalent a posteriori error estimates, parabolic equations, multi-meshes

1. Introduction

For the optimal control problems governed by linear elliptic or parabolic state equations, a priori error estimates of the finite element approximation were established long ago, see [1, 2, 3, 4, 5]. In order to obtain a numerical solution of acceptable accuracy for the optimal control problem, the finite element meshes have to be refined according to a mesh refinement scheme. Adaptive finite element approximation uses a posteriori indicators to guide the mesh refinement procedure. Only the area where the error indicator is larger will be refined so that a higher density of nodes is distributed over the area where the solution is difficult to be approximated. In this sense adaptive finite element approximation relies very much on the error indicator used.

It has been recently found that suitable adaptive meshes can greatly reduce the control approximation errors, see [6, 7, 8, 9, 10]. If the computational meshes are not properly generated, then there may be large error around the singularities of the control, which cannot be removed later on. Furthermore in a constrained control problem, the optimal control and the state usually have very different regularities and their locations. This indicates that the all-in-one mesh strategy may be inefficient. Adaptive multi-meshes, that is, separate adaptive meshes which are adjusted according to different error indicators, are often necessary, see [11]. Using different adaptive meshes for the control and the state allows very coarse meshes to be used in solving the state and co-state equations. Thus much computational work can be saved because one of the major computational loads is to solve the state and co-state equations repeatedly. This can be clearly seen from numerical experiments in [11] and our numerical tests in Section 4.

Received by the editors January 31, 2010 and, in revised form, August 27, 2011.
2000 *Mathematics Subject Classification.* 65N30, 49J20 .
This research was supported by the NSF of China(Nos.10571108,10671113) and the Scientific Research Award Fund for Excellent Middle-Aged and Young Scientists of Shandong Province (No. BS2009NJ003), China.

Up to now, a posteriori error estimates have mainly been developed for elliptic control problems particularly with point-wise type control constraints. The details can be found in the book [12]. In a recent work [13], a posteriori error estimates were derived for the constrained optimal control governed by an elliptic equation, where the control constraint is given in an average sense: $K = \{\int u \geq 0\}$. These estimates were held on different multi-meshes for the control and the state.

Although there are so much progress for elliptic control problems, it is much more complicated to study and implement adaptive multi-meshes computational schemes for evolutional control problems. There are some papers on a posteriori error estimates for optimal control problems governed by parabolic equations, e.g. [10, 11]. They mainly used the well-known stability results of the dual equations [14] to derive a posteriori error upper bounds, which were presented mainly in $C(0,T;L^2(\Omega))$-norm. However, a posteriori error lower bounds were not provided in these papers.

The purpose of this work is to derive equivalent a posteriori error estimates for the following constrained parabolic optimal control problem:

$$\min_{u\in X, u(t)\in K} \frac{1}{2}\int_0^T \left\{ \|y-y_d\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \right\} dt, \tag{1.1}$$

subject to

$$\begin{cases} \dfrac{\partial y}{\partial t} - div(\nabla y) = f + u, & (x,t)\in \Omega\times(0,T], \\ y|_{\partial\Omega} = 0, & t\in[0,T], \\ y(x,0) = y_0(x), & x\in\Omega, \end{cases} \tag{1.2}$$

where: Ω is a bounded open set in R^n $(n \geq 2)$ with the Lipschitz boundary $\partial\Omega$, $y_0 \in H_0^1(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$, $U = L^2(\Omega)$, $X = L^2(0,T;U)$. Let $K = \{u(t) \in L^2(\Omega) : a \leq \int_\Omega u(t) \leq b\}$ be a closed convex set, where a and b are known constants. We obtain a posteriori error estimates with lower and upper bounds and present numerical experiments to confirm the effectiveness of the error estimates.

The plan of the paper is as follows. In Section 2, we will construct the multi-meshes finite element approximation in the backward Euler scheme for (1.1)-(1.2). In Section 3, equivalent a posteriori error estimates are derived for both the state and the control approximation. Our methods are very different from that of [10, 11]. For the reason to derive the lower bounds, we do not use the stability results of the dual equations and derive the error estimates mainly in $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;H^1(\Omega))$-norm (see Theorem 3.1). Finally numerical experiments are presented in Section 4. To our best knowledge, this paper appears to be the first trial to consider this case in the literature.

In this paper we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and seminorm $|\cdot|_{W^{m,q}(\Omega)}$. We set $W_0^{m,q}(\Omega) \equiv \{w \in W^{m,q}(\Omega) : w|_{\partial\Omega} = 0\}$. We denote $W^{m,2}(\Omega)$ $(W_0^{m,2}(\Omega))$ by $H^m(\Omega)$ $(H_0^m(\Omega))$. In addition, c or C denotes a general positive constant independent of h.

2. Finite element approximation

In the rest of the paper, we will take the state space $W = L^2(0,T;V)$ with $V = H_0^1(\Omega)$, the control space $X = L^2(0,T;U)$ with $U = L^2(\Omega)$.

Let

$$a(v,w) = \int_\Omega (\nabla v)\cdot\nabla w, \ \ \forall\, v,w \in V; \qquad (u,v) = \int_\Omega uv, \ \ \forall\, u,\, v \in U.$$