INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 10, Number 1, Pages 221–232

REPRESENTATION OF MATCHED-LAYER KERNELS WITH VISCOELASTIC MECHANICAL MODELS

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Abstract. The Kosloff & Kosloff (KK) absorbing-boundary method is shown to be a particular case of the split-PML method introduced by Bérenger. In its original form, the PML technique has been implemented for Maxwell's electromagnetic equations. On the other hand, the KK method was applied to the Schrödinger and acoustic wave equations. Both techniques have subsequently widely been used in dynamic elasticity, involving different rheological equations, including poroelasticity, and electromagnetism. The coordinate stretching used in the PML method is equivalent to the damping kernel in the KK method, which is based on the Maxwell viscoelastic model. Inside the absorbing strips, the result is a traveling wave which gradually attenuates without changing shape or undergoing dispersion. Moreover, we also show that the recently developed unsplit C-PML method is based on the memory-variable formalism to describe anelasticity introduced by Carcione and co-workers, and that the damping kernel is based on the Zener viscoelastic model. The theoretical reflection coefficients, i.e., before discretization, are obtained and re-interpreted using the theory of viscoelasticity through the acoustic/electromagnetic analogy.

Key words. Absorbing boundaries, viscoelasticity, electromagnetism, reflection of waves.

1. Introduction

The solution of partial differential equations describing several physical processes, mainly related to wave propagation in electromagnetism and dynamic elasticity, are generally solved by using direct methods, based on finite differences, finite elements or pseudospectral methods. In order to avoid reflections and/or wraparound from the edges of the numerical mesh, damping has to be implemented at the boundaries in the form of absorbing strips.

Kosloff and Kosloff [18] introduced a modification of the wave equation inside the absorbing strips, where the solution is a wave traveling without dispersion but whose amplitude decreases with distance at a frequency independent rate. A traveling pulse will thus diminish in amplitude without a change of shape. The method has been applied to the Schrödinger and acoustic wave equations. Subsequenty, this method has been applied to different rheological equations, namely, anisotropy, viscoelasticity and poroelasticity, and to electromagnetism, mainly in algorithms where the spatial derivatives are computed with pseudospectral methods. A review can be found, for instance, in Carcione et al. [9] and Carcione [6]. In particular Carcione et al. [10] and Kosloff et al. [17] applied the method to simulate anelastic wavefields and to the elastic wave equation for modeling surface waves, respectively. They have used the Fourier and Chebyshev pseudospectral operators to compute the spatial derivatives. We note here that the Chebyshev method allows us to use an alternative non-reflecting boundary condition at the edges of the mesh, based on characteristics variables, similar to the paraxial wave equation.

On the other hand, the split-PML method has been proposed by Bérenger as an absorbing boundary condition for electromagnetic waves. The PML method has

Received by the editors February 22, 2011 and, in revised form, December 30, 2011.
2000 *Mathematics Subject Classification.* 65M06, 65M22, 65M70.
This research was supported by the CO2-CARE project.

been widely used for finite-difference and finite-element methods. Chew and Liu [13] first proposed the PML method for elastic waves in solids. A recent implementation can be found in Festa and Vilotte [15], where these authors provide a review of the application of the PML method to different stress-strain relations in elastodynamics.

The similarity between the two methods under a practical choice of the attenuation parameter is shown in this paper. The advantage of the split-PML method is that the exact reflection coefficient is zero at all angles of incidence in contrast to the KK method, whose corresponding reflection coefficient is zero only at normal incidence. However, both coefficients differ from zero after the spatial discretization. As shown by Komatitsch and Martin [16], the split nature of the PML method causes spurious events at grazing angles. It is shown by these authors and by Bérenger [2] that a further improvement is obtained by using the C-PML approach. The so-called unsplit C-PML, introduced by Roden and Gedney [19] in electromagnetism, is based on a convolutional relation similar to the stress-strain convolution used by Carcione et al. [10, 11] to simulate anelastic fields in the time domain. In order to overcome the convolutions, additional variables and therefore additional differential equations have to be introduced in the formulation. It is shown in this work that the differential equations used in the C-PML method are the same to the ones obtained by Carcione et al. [10, 11] to model viscoelasticity with the Zener model. The equivalence is also shown in the frequency domain.

2. The KK method

The boundaries of the numerical mesh may generate non-physical artifacts which disturb the physical events. These artifacts consists in field wraparound when using the Fourier method and reflections when using the Chebyshev pseudospectral method.

When we consider the constant-density pressure formulation, the wave equation can be written as a system of coupled equations and modified as

$$\frac{\partial}{\partial t}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -\gamma & 1 \\ c^2\Delta & -\gamma \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix} \tag{1}$$

where p is the pressure, f are body forces, c is the wave velocity, Δ is the Laplacian, and γ is the absorbing parameter or damping factor. Note that $q = \dot{p}$ if $\gamma = 0$, where the dot above a variable denotes time differentiation. Hereafter, the indices 1, 2 and 3 correspond to the spatial variables x, y and z or x_1, x_2 and x_3, respectively.

The absorbing-boundary parameter $\gamma(x, y, z)$ differs from zero only in a strip of nodes surrounding the numerical mesh. Its spatial dependence is chosen to achieve the best amplitude reduction. The following spatial dependence was chosen in Kosloff and Kosloff [18],

$$\gamma = U_0/\cosh^2(d\ n), \tag{2}$$

where U_0 is a constant, d is a decay factor and n denotes the distance in number of grid points from the boundary. In quantum mechanics, the function γ plays a similar role to a complex negative potential added to the Hamiltonian.

A single second-order equation can be obtained after elimination of the variable q. For instance, in the case of homogeneous media and in the absence of the source we obtain

$$\ddot{p} = c^2\Delta p - 2\gamma\dot{p} - \gamma^2 p. \tag{3}$$