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STABILITY AND DISPERSION ANALYSIS OF THE STAGGERED DISCONTINUOUS GALERKIN METHOD FOR WAVE PROPAGATION

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Abstract. Staggered discontinuous Galerkin methods have been developed recently and are adopted successfully to many problems such as wave propagation, elliptic equation, convection-diffusion equation and the Maxwell's equations. For wave propagation, the method is proved to have the desirable properties of energy conservation, optimal order of convergence and block-diagonal mass matrices. In this paper, we perform an analysis for the dispersion error and the CFL constant. Our results show that the staggered method provides a smaller dispersion error compared with classical finite element method as well as non-staggered discontinuous Galerkin methods.

Key words. CFL condition, dispersion analysis, dispersion relation, wave propagation, staggered discontinuous Galerkin method

1. Introduction

Discontinuous Galerkin method has become a class of very popular, efficient and highly accurate methodologies for the numerical approximation of wave equations [12, 13, 14, 15]. There are many studies in literature regarding their numerical performance as well as stability and convergence analysis. However, dispersion analysis is rarely seen despite its importance for wave propagation. The first attempt to analyze the numerical dispersion for discontinuous Galerkin method for the scalar wave equation has been carried out in [1], where a complete dispersion analysis for the interior penalty, upwind and central discontinuous Galerkin methods are performed for the numerical approximation of the wave equation in both first order and second order forms. Besides, in [11], dispersion analysis for high order discontinuous Galerkin methods applied to three dimensional Maxwell's equations with both centered and uncentered fluxes are carried out. Some superconvergence results on the dispersion error are also obtained in this work.

Recently, staggered discontinuous Galerkin methods have been developed and are adopted successfully to many problems such as wave propagation [2, 3, 4, 5, 6], elliptic equation [7], convection-diffusion equation [9] and the Maxwell's equations [8]. For the numerical simulation of waves, the method is proved to have the desirable properties of energy conservation, optimal order of convergence and block-diagonal mass matrices. Our aims in this paper are to estimate the CFL stability condition corresponding to the leap-frog time discretization and derive the dispersion relation for the staggered discontinuous Galerkin method developed in [3, 4] for wave propagation. We will show that this method has a better CFL number and a smaller dispersion error compared with the classical conforming finite element for second order wave equation [10] as well as the upwind and central discontinuous Galerkin method for wave equation in first order form [1].

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2. The staggered discontinuous Galerkin method

In this section, we will present the staggered discontinuous Galerkin method developed by Chung and Engquist [3, 4] for the numerical simulation of waves. To facilitate the stability and dispersion analysis, we consider the one-dimensional scalar wave equations

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial p}{\partial x}, \tag{1}$$

$$\frac{\partial p}{\partial t} = \frac{\partial u}{\partial x}, \tag{2}$$

for $(x,t) \in (-\infty,\infty) \times [0,\infty)$, where $c > 0$ is the scalar wave speed. Moreover, we will consider a uniform partition. Let $h > 0$ be the mesh size and let $x_j = jh$, $j = 0, \pm 1, \pm 2, \cdots$, be the nodal points. We define the primal cell $I_{j+\frac{1}{2}} = (x_j, x_{j+1})$. For each primal cell $I_{j+\frac{1}{2}}$, we take a point $x'_{j+\frac{1}{2}} \in I_{j+\frac{1}{2}}$ and define the dual cell I'_j by $I'_j = (x'_{j-\frac{1}{2}}, x'_{j+\frac{1}{2}})$. To simplify the analysis, we will take $x'_{j+\frac{1}{2}}$ to be the mid-point of $I_{j+\frac{1}{2}}$, that is, $x'_{j+\frac{1}{2}} = x_{j+\frac{1}{2}}$ and consequently $I'_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$.

Multiplying both sides of (1) by a test function ϕ, integrating on a primal cell $I_{j+\frac{1}{2}}$ and using integration by parts yields

$$\int_{I_{j+\frac{1}{2}}} \frac{\partial u}{\partial t}\phi \, dx = c^2 \Big\{ p(x_{j+1},t)\phi(x_{j+1}) - p(x_j,t)\phi(x_j) - \int_{I_{j+\frac{1}{2}}} p \frac{d\phi}{dx}\, dx \Big\}.$$

Similarly, multiplying both sides of (2) by a test function ψ, integrating on a dual cell I'_k and using integration by parts yields

$$\int_{I'_k} \frac{\partial p}{\partial t}\psi \, dx = u(x_{k+\frac{1}{2}},t)\psi(x_{k+\frac{1}{2}}) - u(x_{k-\frac{1}{2}},t)\psi(x_{k-\frac{1}{2}}) - \int_{I'_k} u \frac{d\psi}{dx}\, dx.$$

The staggered discontinuous Galerkin method can be described as follows. Find $u_h \in U_h$ and $p_h \in W_h$ such that

$$\int_{I_{j+\frac{1}{2}}} \frac{\partial u_h}{\partial t}\phi \, dx = c^2 \Big\{ p_h(x_{j+1},t)\phi(x_{j+1}) - p_h(x_j,t)\phi(x_j) - \int_{I_{j+\frac{1}{2}}} p_h \frac{d\phi}{dx}\, dx \Big\}, \tag{3}$$

$$\int_{I'_k} \frac{\partial p_h}{\partial t}\psi \, dx = u_h(x_{k+\frac{1}{2}},t)\psi(x_{k+\frac{1}{2}}) - u_h(x_{k-\frac{1}{2}},t)\psi(x_{k-\frac{1}{2}}) - \int_{I'_k} u_h \frac{d\psi}{dx}\, dx, \tag{4}$$

for all $\phi \in U_h$ and $\psi \in W_h$, and for all integers j and k.

We will now discuss the choice of the two finite element spaces U_h and W_h. Let $m \geq 0$ be an integer, that corresponds to the degree of polynomials used for trial and test spaces. For each given primal cell $I_{j+\frac{1}{2}}$, we define $R_m(I_{j+\frac{1}{2}})$ as the space of functions which are polynomials of degree at most m on each of the two sub-cells $(x_j, x_{j+\frac{1}{2}})$ and $(x_{j+\frac{1}{2}}, x_{j+1})$ with continuity at $x_{j+\frac{1}{2}}$. Similarly, for each given dual cell I'_k, we define $R'_m(I'_k)$ as the space of functions which are polynomials of degree at most m on each of the two sub-cells $(x_{k-\frac{1}{2}}, x_k)$ and $(x_k, x_{k+\frac{1}{2}})$ with continuity at x_k. We will state the definitions of U_h and W_h in the following.

Definition 1. *The two finite element spaces U_h and W_h are defined by*

(1) $\phi \in U_h$ *if* $\phi|_{I_{j+\frac{1}{2}}} \in R_m(I_{j+\frac{1}{2}})$.
(2) $\psi \in W_h$ *if* $\psi|_{I'_k} \in R'_m(I'_k)$.

In Figure 1, typical functions in the spaces U_h and W_h are shown for the piecewise linear case, that is $m = 1$. Here, we use solid line to represent a function in U_h and