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ERROR ESTIMATES FOR THE SECOND ORDER SEMI-DISCRETE STABILIZED GAUGE-UZAWA METHOD FOR THE NAVIER-STOKES EQUATIONS

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Abstract. The Gauge-Uzawa method [GUM], which is a projection type algorithm to solve the time depend Navier-Stokes equations, has been constructed in [14] and enhanced in [15, 17] to apply to more complicated problems. Even though GUM possesses many advantages theoretically and numerically, the studies on GUM have been limited on the first order backward Euler scheme except normal mode error estimate in [16]. The goal of this paper is to research the 2nd order GUM. Because the classical 2nd order GUM which is studied in [16] needs rather strong stability condition, we modify GUM to be unconditionally stable method using BDF2 time marching. The stabilized GUM is equivalent to the rotational form of pressure correction method and the errors are already estimated in [8] for the Stokes equations. In this paper, we will evaluate errors of the stabilized GUM for the Navier-Stokes equations. We also prove that the stabilized GUM is an unconditionally stable method for the Naiver-Stokes equations. So we conclude that the rotational form of pressure correction method in [8] is also unconditionally stable scheme and that the accuracy results in [8] are valid for the Navier-Stokes equations.

Key Words. Projection method, Gauge-Uzawa method, the rotational form of pressure correction method, Navier-Stokes equations, incompressible fluids

1. Introduction

Given an open bounded polyhedral domain Ω in $\mathbb{R}^d$, with $d = 2$ or 3, we consider the time-dependent Navier-Stokes equations of incompressible fluids:

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \mu\triangle\mathbf{u} &= \mathbf{f}, && \text{in } \Omega,\\ \nabla\cdot\mathbf{u} &= 0, && \text{in } \Omega,\\ \mathbf{u}(0,\mathbf{x}) &= \mathbf{u}^0, && \text{in } \Omega, \end{aligned} \tag{1.1}$$

with vanishing Dirichlet boundary condition $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$ and pressure mean-value $\int_\Omega p = 0$. The primitive variables are the (vector) velocity $\mathbf{u}$ and the (scalar) pressure p. The viscosity $\mu = Re^{-1}$ is the reciprocal of the Reynolds number Re. Hereafter, vectors are denoted in boldface.

Pressure p can be viewed in (1.1) as a Lagrange multiplier corresponding to the incompressibility condition $\nabla\cdot\mathbf{u} = 0$. This coupling is responsible for compatibility conditions between the spaces for $\mathbf{u}$ and p, characterized by the celebrated inf-sup condition, and associated numerical difficulties [5, 20]. On the other hand, projection methods were

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introduced independently by Chorin [1] and Temam [19] in the late 60's to decouple $\mathbf{u}$ and p and thus reduce the computational cost. And the methods quickly gained popularity in the computational fluid dynamics community, and over the years, an enormous amount of efforts have been devoted to develop more accurate and efficient projection type schemes, we refer to [16, 7] for comprehensive and up-to-date review on this subject.

Because most engineers prefer higher order methods, many projection methods have been built using 2nd order time discrete schemes which are the Crank-Nicolson scheme and 2nd order backward difference formulation [BDF2]. In general, both of them have same accuracy, but BDF2 displays better numerical behaviors on stability than the Crank-Nicolson scheme. So most of new methods, the pressure-correction in [6], the velocity-correction in [10, 3], and the consistent splitting method in [9] have been studied with respect to BDF2 for time. In addition, the rotational form of pressure-correction method has been introduced in [21] with embarking BDF2 and then the errors have been evaluated via energy estimate in [8] and via normal mode analysis in [16] for the Stokes equations. We also study the method in Algorithm 2 below and discuss about the difficulty to control non-linear term in Remark 1 below. One of the goal of this paper is to extend the accuracy results to the Navier-Stokes equations.

On the other direction, the Gauge-Uzawa method [GUM] has been constructed in [14] to solve (1.1) and enhanced to solve more complicated problems which are the Boussinesq equations in [15] and the non-constant density fluid problems in [17]. However, GUM has been studied only for the 1st order backward Euler scheme for time except normal mode error analysis in [16]. The goal of this paper is to research for the BDF2 GUM to solve Navier-Stokes equations. The classical GUM in [16] displays superior numerical behavior on accuracy, but the method requires rather strong stability condition. In [16, 7], it is known that the classical GUM is a equivalent to the consistent splitting scheme in [9]. So both methods request high computational cost due to tiny τ to make hold the stability constraint. In this paper, we newly construct the stabilized BDF2 GUM and prove optimal error estimates for the Navier-Stokes equations. We will also prove that the method is unconditionally stable scheme for any time step τ. In addition, we discover that the stabilized BDF2 GUM is equivalent to the rotational form of pressure correction method in [8]. So we conclude that the rotational form of pressure correction method is also unconditionally stable for any τ and that the error decay results in [8] are extended to the Navier-Stokes equations.

In this paper, we will use standard notations. Let $H^s(\Omega)$ be the Sobolev space with s derivatives in $L^2(\Omega)$, $\mathbf{L}^2(\Omega) = \left(L^2(\Omega)\right)^d$ and $\mathbf{H}^s(\Omega) = \left(H^s(\Omega)\right)^d$, where $d = 2, 3$. Let $\|\cdot\|_0$ denote the $\mathbf{L}^2(\Omega)$ norm, and $\langle\cdot\,,\,\cdot\rangle$ the corresponding inner product. Let $\|\cdot\|_s$ denote the norm of $H^s(\Omega)$ for $s \in \mathbb{R}$. In addition, we will denote τ as the time marching size. Also we will use δ as difference of 2 consecutive functions, for example, for any sequence function z^{n+1},

$$\delta z^{n+1} = z^{n+1} - z^n, \quad \delta\delta z^{n+1} = \delta(\delta z^{n+1}) = z^{n+1} - 2z^n + z^{n-1}, \quad \cdots .$$

This paper is organized as follows. In §2, we will derive the 2nd order stabilized GUMs and the rotational form of pressure correction method in [8]. And then we state main theorems for stability and accuracy. We introduce some well-known lemma in §3 to use in theoretical proofs. We then prove stability of the stabilized GUM in §4 and estimate errors of the stabilized GUM in §5. We finally conclude in §6 with numerical tests to compare with theoretical results.

2. The stabilized Gauge-Uzawa method

In this section. we will derive the stabilized BDF2 time discrete GUM and the rotational form of projection method in [8, 16]. We will conclude that both of them are