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MULTISCALE COMPUTATION OF A STEKLOV EIGENVALUE PROBLEM WITH RAPIDLY OSCILLATING COEFFICIENTS

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Abstract. In this paper we consider the multiscale computation of a Steklov eigenvalue problem with rapidly oscillating coefficients. The new contribution obtained in this paper is a superapproximation estimate for solving the homogenized Steklov eigenvalue problem and to present a multiscale numerical method. Numerical simulations are then carried out to validate the theoretical results reported in the present paper.

Key Words. Steklov eigenvalue problem, multiscale method, superapproximation estimate.

1. Introduction

In this paper we discuss the multiscale computation of a Steklov eigenvalue problem with rapidly oscillating coefficients given by

$$\begin{cases} \mathcal{L}_\varepsilon u^\varepsilon = 0, & \text{in} \quad \Omega, \\ u^\varepsilon = 0, & \text{on} \quad \Gamma_0, \\ \sigma_\varepsilon(u^\varepsilon) = \lambda^\varepsilon u^\varepsilon, & \text{on} \quad \Gamma_1, \end{cases} \tag{1}$$

where Ω is a bounded Lipschitz polygonal convex domain or a smooth domain in R^n, $n \geq 2$ with a periodic microstructure, and whose boundary is denoted by $\Gamma = \partial\Omega = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, with $\Gamma_0 \cap \Gamma_1 = \emptyset$. Here $\mathcal{L}_\varepsilon$ denotes a second-order partial differential operator with rapidly oscillating coefficients given by

$$\mathcal{L}_\varepsilon \phi \equiv -\frac{\partial}{\partial x_i}\Big(a_{ij}(\frac{x}{\varepsilon})\frac{\partial \phi}{\partial x_j}\Big) + a_0(\frac{x}{\varepsilon})\phi,$$

and

$$\sigma_\varepsilon(\phi) \equiv \nu_i a_{ij}(\frac{x}{\varepsilon})\frac{\partial \phi}{\partial x_j},$$

where $\vec{\nu} = (\nu_1, \cdots, \nu_n)$ is the outward unit normal to Γ_1, and $\varepsilon > 0$ is a small period parameter. Here and below we use the Einstein summation convention on repeated indices.

We make the following assumptions:

$(\mathbf{A_1})$ Let $\xi = \varepsilon^{-1}x$, and assume that $a_{ij}(\xi)$, $a_0(\xi)$ are 1-periodic functions in ξ.

$(\mathbf{A_2})$ There is a positive constant γ_0 which is independent of ε such that

$$a_{ij}(\frac{x}{\varepsilon})\eta_i\eta_j \geq \gamma_0|\eta|^2$$

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for all $(\eta_1, \cdots, \eta_n) \in R^n$, $|\eta|^2 = \sum_{i=1}^{n} \eta_i^2$ and all $x \in \Omega$, $a_0(\frac{x}{\varepsilon}) \geq 0$.

$(\mathbf{A_3})$ $a_{ij}(\frac{x}{\varepsilon}) = a_{ji}(\frac{x}{\varepsilon})$ for almost every $x \in \Omega$.

$(\mathbf{A_4})$ $a^{\varepsilon}_{ij}, a^{\varepsilon}_0 \in L^{\infty}(\overline{\Omega})$.

Problems with an eigenvalue parameter on the boundary appear in many physical situations(see, e.g.[1, 10, 26, 3]). Courant and Hilbert [16] presented early results on the Steklov eigenvalue problems. Osborn [36]developed a general approximation theory for compact operators. Bramble and Osborn [7] presented a Galerkin method for the approximation of the Steklov problem for a non self-adjoint second order differential operator. Andreev and Todorov [2] gave the isoparametric finite element approximation of Steklov eigenvalue problems for second-order, self-adjoint, elliptic differential operators. Several eigenvalue problems arising in physics and engineering, as well as their approximations, are presented in Weinberger [42],Babuska and Osborn [5]. On the other hand, a Steklov eigenvalue problem with constant coefficients can be easily converted into the eigenvalue problem of a boundary integral equation, so the boundary element method is more advantageous in such a case. Han, Guan and He [24] developed the boundary element method for a Steklov eigenvalue problem by means of a boundary integral equation. Huang and Lü [29] used the mechanical quadrature method to obtain the extrapolation formulae for solving the boundary integral equation arising from Steklov eigenvalue problems.

This paper involves Steklov eigenvalue problems arising from structures made of composite materials. In such cases, the direct accurate numerical computation of the solution becomes difficult because of the very fine mesh required. We recall that the homogenization method gives the overall solution behavior by incorporating the fluctuations due to the heterogeneities. Vanninathan [40] obtained a homogenization result for a spectral problem with Steklov boundary conditions on periodically distributed holes inside the domain Ω. There are many other results (see, e.g. [33, 21, 23, 30, 9]). Simulation results (cf. [11], [12]) have shown that the numerical accuracy of the homogenization method may not be satisfactory if ε is not sufficiently small. We also refer to the numerical results presented in Section 5. This is the main motivation for the multiscale asymptotic methods and the associated numerical algorithms.

Sarkis and Versieux [41] presented the numerical boundary corrector for elliptic equations with rapidly oscillating periodic coefficients and derived the convergence results of their method in [39]. Hou and Wu[27] and Hou, Wu and Cai [28] provided an interesting multiscale finite element method (MsFEM) based on the first order asymptotic expansion. The basic idea of MsFEM is to find new finite element space; i.e., the set of basis functions consists of two parts, the first part being the set of piecewise polynomials and the second part the set of some oscillatory functions obtained by simultaneously solving locally partial differential equations in subdomains. Efendiev and Hou [19] gave a comprehensive survey of MsFEM. E and Engquist [17] proposed the overall framework of an important heterogeneous multi-scale method (HMM). A review of HMM was presented in [18]. In [13], authors presented recently the multiscale asymptotic method for a Steklov problem with rapidly oscillating coefficients.

The new contribution obtained in the present paper is a superapproximation estimate for solving the homogenized Steklov eigenvalue problem and to present a multiscale finite element method on the basis of the theoretical results of [13]. The key steps in this approach are the application of an adaptive finite element