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# UNCONDITIONAL CONVERGENCE OF HIGH-ORDER EXTRAPOLATIONS OF THE CRANK-NICOLSON, FINITE ELEMENT METHOD FOR THE NAVIER-STOKES EQUATIONS

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**Abstract.** Error estimates for the Crank-Nicolson in time, Finite Element in space (CNFE) discretization of the Navier-Stokes equations require application of the discrete Gronwall inequality, which leads to a time-step ($\Delta t$) restriction. All known convergence analyses of the fully discrete CNFE with linear extrapolation rely on a similar $\Delta t$-restriction. We show that CNFE with arbitrary-order extrapolation (denoted CNLE) is *convergences optimally in the energy norm* without any $\Delta t$-restriction. We prove that CNLE velocity and corresponding discrete time-derivative converge optimally in $l^\infty(H^1)$ and $l^2(L^2)$ respectively under the mild condition $\Delta t \leq Mh^{1/4}$ *for any arbitrary* $M > 0$ (e.g. independent of problem data, $h$, and $\Delta t$) where $h > 0$ is the maximum mesh element diameter. Convergence in these higher order norms is needed to prove convergence estimates for pressure and the drag/lift force a fluid exerts on an obstacle. Our analysis exploits the extrapolated convective velocity to avoid any $\Delta t$-restriction for convergence in the energy norm. However, the coupling between the extrapolated convecting velocity of usual CNLE and the *a priori* control of *average* velocities (characteristic of CN methods) rather than pointwise velocities (e.g. backward-Euler methods) in $l^2(H^1)$ is precisely the source of $\Delta t$-restriction for convergence in higher-order norms.

**Key words.** Navier-Stokes, Crank-Nicolson, finite element, extrapolation, linearization, error, convergence, linearization

## 1. Introduction

The usual Crank-Nicolson (CN) in time Finite Element (FE) in space discretization of the Navier-Stokes (NS) Equations (NSE) denoted by CNFE is well-known to be unconditionally (energetically) stable. The error analysis of the CNFE method is based on a discrete Gronwall inequality which introduces a time-step ($\Delta t > 0$) restriction (for convergence, not for stability) of the form

$$\Delta t \leq C(Re, h), \qquad \text{e.g.} \quad \Delta t \leq \mathcal{O}(Re^{-3}) \tag{1}$$

(see Appendix A for a derivation, Theorem A.1 with e.g. (157)). Here $h > 0$ is the maximum mesh element diameter and $Re > 0$ is the Reynolds number. Condition (1)(a) implies *conditional convergence* whereas (1)(b) is a *robustness condition* and both are prohibitively restrictive in practice; for example, (1)(b) suggests

$$Re = 100 \text{ (low-to-moderate value)} \quad \Rightarrow \quad \Delta t \leq \mathcal{O}(10^{-6}).$$

Consequently, an important open question regards whether condition (1) is

- an artifact of imperfect mathematical technique, or
- a special feature of the CN time discretization.

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We consider the necessity of a $\Delta t$-restriction in a linear, fully implicit variant of CNFE obtained by extrapolation of the convecting velocity $\mathbf{u}$: for example, suppressing spatial discretization, given $\mathbf{u}^0$, $\mathbf{u}^1$, and $p^1$, for each $n = 1, 2, \ldots$ find velocity $\mathbf{u}^{n+1}$ and pressure $p^{n+1}$ satisfying

$$\begin{aligned} &\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + (\frac{3}{2}\mathbf{u}^n - \frac{1}{2}\mathbf{u}^{n-1}) \cdot \nabla \frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2} \\ &\quad - Re^{-1} \Delta \frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2} + \nabla \frac{p^{n+1} + p^n}{2} = \frac{\mathbf{f}^{n+1} + \mathbf{f}^n}{2} \\ &\nabla \cdot \mathbf{u}^{n+1} = 0. \end{aligned} \tag{2}$$

Here $\mathbf{f}$ is body-force term, and $z^i := z(x, t^i)$ and $t^i = i\Delta t$. This method is often called CNLE and was first studied by Baker [3]. CNLE is linearly implicit, unconditionally (energetically) stable, and second-order accurate. In this report, we show that *no $\Delta t$-restriction* is required for the convergence of CNLE (Proposition 3.1, Theorem 3.5). In particular,

$$||error(CNLE)||_{l^\infty(L^2)\cap l^2(H^1)} \leq C(h^k + \Delta t^2), \qquad k = \text{ degree of FE space}$$

(Theorem 3.5). This result was proved for the semi-discrete case as $\Delta t \to 0$ in [10] and the fully discrete Backward Euler (BE) scheme with Constant Extrapolation (BECE) in [32]. The analysis depends on

- *Gronwall inequality* - exploit time-lagged convecting velocity (Section 1.1)
- Estimate (74) - bound convecting velocity in $L^2$ (Section 1.1.1)

Indeed, for extrapolated CN, we apply a discrete Gronwall Lemma without any $\Delta t$-restriction; for general extrapolations we derive and apply the estimate (74)(b) of the *explicitly* skew-symmetric convective term. We explain our strategy for proving the CNLE error estimate and corresponding difficulties in detail in Section 1.1.

We also prove convergence estimates in higher-order norms. We show that the CNLE velocity approximation converges optimally in the $l^\infty(H^1)$-norm and the corresponding discrete time-derivative of the velocity approximation converges optimally in the $l^2(L^2)$-norm (Theorems 3.8, 3.10) under a modest $\Delta t$-restriction

$$\Delta t \leq M h^{1/4}, \quad \text{for any } M > 0 \text{ (no } Re\text{-dependence).} \tag{3}$$

Note that $M$ is completely arbitrary so that (3) only governs the rate at which $\Delta t \to 0$ and not the size of $\Delta t$. In particular, restriction (3) is not a typical artifact of the discrete Gronwall inequality since it does not depend problem data. The error estimate is obtained by a bootstrap argument that utilizes the error in the energy norm. Although such estimates have been proved for BECE in [32], the analysis of CNLE is distinctly different because CN methods only give *a priori* control of *average* velocities $\mathbf{u}^{n+1/2}$ rather than pointwise velocities $\mathbf{u}^{n+1}$ (e.g. BE methods) in $l^2(H^1)$. Our analysis depends on

- Estimate (75) - bound test-function of convective term in $L^2$ (Section 1.1.1)
- CN *a priori* estimates - introduce $\Delta t$-restriction (3) (Section 1.1.2)
- Stokes projection - preserve optimal convergence rate (Section 1.1.3).

Indeed, we derive and apply estimate (75)(b) of the *explicitly* skew-symmetric convective term; we obtain intermediate estimates in the convergence analysis of CNLE with limited options corresponding to limited control of *average* velocities (characteristic of CN methods) in $l^2(H^1)$; and we exploit the Stokes projection to preserve the optimal convergence rate for the FE and CN discretization.