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SEMI-ANALYTICAL NUMERICAL METHODS FOR CONVECTION-DOMINATED PROBLEMS WITH TURNING POINTS

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Abstract. In this article we aim to study finite volume approximations which approximate the solutions of convection-dominated problems possessing the so-called interior transition layers. The stiffness of such problems is due to a small parameter multiplied to the highest order derivative which introduces various transition layers at the boundaries and at the interior points where certain compatibility conditions do not meet. Here, we are interested in resolving interior transition layers at turning points. The proposed semi-analytic method features interior layer correctors which are obtained from singular perturbation analysis near the turning points. We demonstrate this method is efficient, stable and it shows 2^{nd}-order convergence in the approximations.

Key words. Convection-diffusion equations, Singular perturbation analysis, Transition layers, Boundary layers, Compatibility conditions, Turning points, Finite volume methods.

1. Introduction

In this article, we consider a singularly perturbed problem presenting a turning point, that is

$$\begin{cases} L_\epsilon u := -\epsilon u_{xx} - b u_x = f & \text{in } \Omega = (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \tag{1.1}$$

where $0 < \epsilon << 1$, $b = b(x)$, $f = f(x)$ are smooth on $[-1, 1]$, and for $\delta > 0$, $b < 0$, for $-\delta < x < 0$, $b(0) = 0$, $b > 0$ for $0 < x < \delta$, and $b_x(0) > 0$.

In our previous work [16], a new numerical approximation to solve a boundary layer problem, i.e., Eq. (1.1) with the sign of $b(x)$ unchanged, is developed and implemented based on enriched subspace techniques. In this paper, we continue to investigate a more challenging problem possessing an interior transition layer, which is displayed near $x = 0$ where the convective coefficient $b(x)$ changes sign. The point $x = 0$ is called a turning point. The asymptotic analysis for problem (1.1) is fully detailed in [14] depending on the compatibility between b and f.

Transition layers are very thin regions, i.e., their thickness is in the order of the small parameter ϵ, where values of the derivative (or gradient in higher dimensional problems) are much larger than those in outer regions of the solution. They appear in the solution when there is a small parameter multiplying the highest derivative and the coefficient of the convective term changes its sign at points called turning points. Transition layers match the discrepancies between outer solutions which occur at turning points. Thus, in the limit case, i.e. when $\epsilon = 0$, singularity happens in the solution around these turning points. This type of singularity is called asymptotic singularity (see e.g. [10], [12] and [26]). Transition layers interpret significant physical phenomena, for instance, turbulent boundary layers occurring

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at the points where the turbulent boundary layer separates since the tangential velocity vanishes and changes sign at such points (see [3]) in fluid dynamics; or the propagation of light in a nonhomogeneous medium as an application of Maxwell's equations in Electromagnetism (see [1]).

It is well-known that constructing numerical methods for a problem of the type (1.1) is difficult and computationally expensive. It is because very fine meshes are required for the transition layer so that sharp changes in the layer can be accurately captured as well as oscillations due to asymptotic singularity must be prevented from occurring. There are many works devoted to studying the problem, both analytically (see e.g., [8], [9], [14], [17], [23]) and numerically (see e.g., [13], [16], [4], [5], [18]).

Our aim in this article is to construct an accurate and efficient numerical approximation for the solution of Eq. (1.1) based on the technique developed in [16] and the novel asymptotic analysis as in [14]. The advantage of our scheme is that the solution of the transition layer is resolved analytically, thus the mesh size does not rely on the thickness of the layer, leading to a much reduction in computational cost but still preserving the properties of a good approximation mentioned above. The extension to more complex problems, e.g., multiple transition layers, transition layers incorporating boundary layers, or the coefficient $b(x)$ having zeros with multiplicity, etc. is discussed in the Conclusion and will appear in subsequent papers.

The article is organized as follows. In section 2.1, we introduce a conventional finite volume scheme for problem (1.1) (see e.g., [25]). We then employ singular perturbation analysis (see e.g., [9], [14], [19], [20], [22]) in order to derive the exact solution for the transition layer of problem (1.1) in section 2.2. Based on this, a new Finite Volume discretization is introduced in section 2.3. Numerical results illustrating for the methods are presented in section 3. Finally, we close the article with the Conclusion section.

2. Discretizations

To approximate the solution of Eq. (1.1), we employ Finite Volume discretizations. We first introduce a classical finite volume method and then, via singular perturbation analysis, we derive some transition layer correctors which capture the transition layer and spikes caused by the noncompatibility in the data of Eq. (1.1) (see section 2.3.2 below). These correctors are incorporated in the classical scheme to produce a stable, accurate and efficient scheme.

2.1. Classical Finite Volume Method (cFVM). In this section, we apply finite volume discretizations in approximating the solution of Eq. (1.1). Firstly, we define the mesh parameters for our scheme. We rather use a uniform mesh for our computation. Let x_j, u_j be nodal points and values, respectively. The x_j are located at $x = -1 + (j - 1/2)h,\ h = 2/N,\ j = 0, 1, 2, \dots, N, N + 1$ where h is the mesh size and N is the number of control volumes. The points $x_0,\ x_{N+1}$ are called ghost points or fictitious points which do not belong to the computational domain Ω and their nodal values $u_0,\ u_{N+1}$ are determined via boundary conditions and appropriate interpolations at the boundaries (see (2.6) and (2.7) below). Then the control volumes at x_j have faces at $x_{j-\frac{1}{2}} = x_j - h/2,\ x_{j+\frac{1}{2}} = x_j + h/2,\ j = 1, 2, \dots, N$. Note that the boundary points are $x_{\frac{1}{2}} = -1,\ x_{N+\frac{1}{2}} = 1$.