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NUMERICAL INVESTIGATION OF THE DECAY RATE OF SOLUTIONS TO MODELS FOR WATER WAVES WITH NONLOCAL VISCOSITY

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Abstract. In this article, we investigate the decay rate of the solutions of two water wave models with a nonlocal viscous term written in the KdV form

$$u_t + u_x + \beta u_{xxx} + \frac{\sqrt{\nu}}{\sqrt{\pi}} \int_0^t \frac{u_t(s)}{\sqrt{t-s}} ds + u u_x = \nu u_{xx}.$$

and

$$u_t + u_x - \beta u_{txx} + \frac{\sqrt{\nu}}{\sqrt{\pi}} \int_0^t \frac{u_t(s)}{\sqrt{t-s}} ds + u u_x = \nu u_{xx}.$$

in the BBM form. In order to realize this numerical study, a numerical scheme based of the G^α-scheme is developed.

Key words. waterwaves, viscous asymptotical models, long-time asymptotics, fractional derivatives.

1. Introduction

Modeling the effect of viscosity on the gravity waves is a challenging issue and much research on this subject has been carried out during last decade. After the pionneer work of Kakutani and Matsuuchi [14], P. Liu and T. Orfila [15], and D. Dutykh and F. Dias [12] have derived, independently, asymptotical models for long gravity waves on viscous shallow water. These models are Boussinesq type systems with a non local in time viscous terms. A one-way reduction of these models was adressed in [11].

Computing the decay rate for solutions of that type of problem is also a challenging issue [1, 3, 4, 8]. In a previous work [7], Chen et *al.* were concerned with computing both theoretically and numerically the decay rate of solutions to a water wave model with a nonlocal viscous dispersive term. This model is the following

$$u_t + u_x + \beta u_{xxx} + \frac{\sqrt{\nu}}{\sqrt{\pi}} \int_0^t \frac{u_t(s)}{\sqrt{t-s}} ds + \gamma u u_x = \alpha u_{xx}, \tag{1}$$

where u is the horizontal velocity of the fluid. This equation requires some comments: the usual diffusion is $-\alpha u_{xx}$, while βu_{xxx} is the geometric dispersion and $\frac{\sqrt{\nu}}{\sqrt{\pi}} \int_0^t \frac{u_t(s)}{\sqrt{t-s}} ds$ stands for the nonlocal diffusive-dispersive term and models the viscosity. Here α, β, γ and ν are non negative parameters dedicated to balance or unbalance the effects of viscosity and dispersion versus the nonlinear effects. Specifically the authors have obtained the following global existence and decay results

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for the problem (1) with $\beta = 0$ (see also [13] for $\beta = 1$ and $\gamma = \alpha = 0$) with small initial datum. More precisely, they state the following theorem.

Theorem 1 (Chen et al., 2009)**.** *Consider* (1) *with* $\beta = 0$ *supplemented with initial data* $u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$*. There exists* $\epsilon > 0$, $C(u_0) > 0$ *such that for all* $\|u_0\|_{L^1(\mathbb{R})} < \epsilon$*, there exists a unique global solution* $u \in C(\mathbb{R}_+; L^2_x(\mathbb{R})) \cap C^1(\mathbb{R}_+; H_x^{-2}(\mathbb{R}))$*. In addition,* u *satisfies*

$$t^{\frac{1}{2}} \|u(t)\|_{L^\infty_x(\mathbb{R})} + t^{\frac{1}{4}} \|u(t)\|_{L^2_x(\mathbb{R})} \leqslant C(u_0) \tag{2}$$

and u *solves the fixed point equation*

$$u(t,x) = K(t,.) \star u_0 + N * u^2, \tag{3}$$

where K *and* N *are given by*

$$K(t,x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} e^{-x^-} \left(1 + \frac{1}{2}\int_0^{+\infty} e^{-\frac{\mu^2}{4t} - \frac{\mu|x|}{2t} - \frac{\mu}{2}} \, d\mu\right)$$

and

$$N(t,x) = \frac{1}{2\sqrt{\pi t}} \partial_x \left[e^{-\frac{x^2}{4t} - x^-} \left(1 - \frac{1}{2}\int_0^{+\infty} e^{-\frac{\mu^2}{4t} - \frac{\mu|x|}{2t} - \frac{\mu}{2}} \, d\mu\right)\right],$$

with $x^- = \max(-x, 0)$*,* $\star$ *denotes the usual convolution product in space and* $*$ *the time-space convolution product defined by*

$$v * w(t,x) = \int_0^t \int_{\mathbb{R}} v(s,y)\, w(t-s, x-y)\, dx\, dy$$

whenever the integrals make sense.

The proof of this theorem can be found in [7].

We also consider the following equivalent BBM (Benjamin-Bona-Mahony) form of the equation (1)

$$u_t + u_x - \beta u_{txx} + \frac{\sqrt{\nu}}{\sqrt{\pi}} \int_0^t \frac{u_t(s)}{\sqrt{t-s}} ds + \gamma u u_x = \alpha u_{xx}. \tag{4}$$

In this article we investigate the asymptotical decay rate of the solutions with several numerical simulations for the two asymptotic models (1) and (4). First, we will compare our numerical simulations to the results of theorem 1 and those from [7] in order to validate the numerical scheme developed in this article. Then, we will discuss in the sequel the role of respectively the non local viscous terms, the geometric dispersion and the nonlinearity.

This article is organized as follows. In the second section, we recall some definitions and give some notations used in this article, such as the Fourier transform and the Gear operator which will be used to approximate the non local viscous term $\frac{\sqrt{\nu}}{\sqrt{\pi}} \int_0^t \frac{u_t(s)}{\sqrt{t-s}} ds$. In section 3, after a presentation of the numerical scheme, we perform several numerical simulations for equation (1). In the last section, we numerically analyze the decay rate of the solutions for equations (4) with different values of the parameters (α, β, γ and ν).

2. Some notations and definitions

2.1. Notations. Let us introduce some notations that we shall use in the sequel. The Fourier transform of a function u in $L^1(\mathbb{R})$ reads

$$\hat{u}(\xi) = \mathcal{F}(u)(\xi) = \int_{\mathbb{R}} u(x)\, e^{-ix\xi}\, dx.$$