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CONVERGENCE OF ROTHE SCHEME FOR HEMIVARIATIONAL INEQUALITIES OF PARABOLIC TYPE

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Abstract. This article presents the convergence analysis of a sequence of piecewise constant and piecewise linear functions obtained by the Rothe method to the solution of the first order evolution partial differential inclusion $u'(t)+Au(t)+\iota^*\partial J(\iota u(t)) \ni f(t)$, where the multivalued term is given by the Clarke subdifferential of a locally Lipschitz functional. The method provides the proof of existence of solutions alternative to the ones known in literature and together with any method for underlying elliptic problem, can serve as the effective tool to approximate the solution numerically. Presented approach puts into the unified framework known results for multivalued nonmonotone source term and boundary conditions, and generalizes them to the case where the multivalued term is defined on the arbitrary reflexive Banach space as long as appropriate conditions are satisfied. In addition the results on improved convergence as well as the numerical examples are presented.

Key words. hemivariational inequality, Rothe method, convergence, existence.

1. Introduction

Partial differential inclusions with the multivalued term given in the form of Clarke subdifferential are known as hemivariational inequalities (HVIs). HVIs are the natural generalization of the inclusions with monotone multivalued term (which lead to variational inequalities) and were firstly considered by Panagiotopoulos in early 1980s. For the description of the origins of HVIs and underlying mathematical theory we refer the reader to the book [30].

This paper deals with the first order evolution inclusion of type $u'(t) + A(u(t)) + \iota^*\partial J(\iota u(t)) \ni f(t)$. Such problems are known as parabolic HVIs or boundary parabolic HVIs depending whether an operator ι is the embedding operator from $H^1(\Omega)$ to $L^2(\Omega)$ or the trace operator from $H^1(\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$. The first case corresponds to multivalued and nonmonotone source term in the equation and the second one to multivalued and nonmonotone boundary conditions of Neumann-Robin type. Such inclusions are used to model the diffusive transport through semipermeable membranes where the multivalued term represents the semipermeability relation [25] and the temperature control problems where the multivalued term represents the feedback control [16], [15].

The existence of solutions to problems governed by inclusions of considered type was investigated by many authors. There are several techniques used to obtain the existence results:

- Classical Faedo-Galerkin approach combined with the regularization of the multivalued term by means of a standard mollifier; solutions of underlying

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system of ordinary differential equations are proved to converge (in appropriate sense) to the function which is shown to be the solution of analyzed HVI. This technique was used in context of parabolic HVIs by Miettinen [23], Miettinen and Panagiotopoulos [25] and Goeleven et al. [14].
- The approach based on the notion of upper and lower solutions. The solution is shown to be the limit of solutions of problems governed by the equations obtained by the regularization of the multivalued term together with the truncation by the lower and upper solutions. The distinctive feature of this approach is that the growth conditions on the multivalued term are replaced by the assumption of the existence of lower and upper solutions. The technique was used for parabolic HVIs by Carl [4] and developed in [5], [6], [7], [9].
- The technique based on showing that the analyzed HVI satisfies the assumptions of the general framework for which the appropriate surjectivity result holds. This approach was used by Liu [20] and by Migórski [27] and developed for the boundary case in [28].
- The technique based on adding to the inclusion the regularizing term multiplied by $\epsilon > 0$, showing that the solutions to obtained problems satisfy some bounds uniformly in ϵ and passing to the limit $\epsilon \to 0$. This technique was used for parabolic HVIs by Liu and Zhang [18] and Liu [19] and developed in [21], [22].

It should be remarked that above techniques are either nonconstructive (i.e. they are based on surjectivity result) or constructive but not effective (i.e. require a priori knowledge of lower and upper solutions, or require additional or smoothing terms in the problem).

In contrast to the existence theory, numerical methods to approximate effectively the solutions to parabolic HVIs were not considered by many authors. In the book of Haslinger, Miettinen and Panagiotopoulos [16] the convergence of solutions obtained by the finite element approximation of the space variable and finite difference approximation of the time variable is proved. However only the case of the linear operator A and the multivalued source term (and not boundary conditions) is considered (see Remark 4.10 in [16]). In [15] the authors proved the convergence of the finite difference scheme (with respect to both time and space variable) for the case of multivalued source term (i.e. $U = H$ in the sequel).

Our approach uses the so-called Rothe method (known also as time approximation method) and allows to extend any numerical method that is used to solve the stationary, elliptic inclusions with the multivalued term given as the Clarke subdifferential, to time dependent, parabolic problems. The key idea is the replacement of time derivative with the backward difference scheme and solve the associated elliptic problem in every time step to find the solution in the consecutive points of the time mesh. It is proved that the results obtained by such approach approximate the solution of the original problem.

On the other hand, the Rothe metod provides the proof of existence of solutions. In contrast to other approaches this metod, as long as one can solve underlying elliptic problems, does not require any smoothing or other additional regularizing terms in the inclusion. Furthermore the presented approach allows to study the inclusions with multivalued term given on the domain and on the domain boundary within the unified framework in which the multifunction that appears in the problem is defined on an arbitrary reflexive Banach space, which satisfies the appropriate assumption ($H(U)$ in the sequel). This assumption is proved to generalize the case of inclusions