INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 10, Number 2, Pages 466–480

HIERARCHICAL A POSTERIORI RESIDUAL BASED ERROR ESTIMATORS FOR BILINEAR FINITE ELEMENTS

MALTE BRAACK AND NICO TASCHENBERGER

Abstract. We present techniques of a posteriori error estimation for Q_1 finite element discretizations based on residual evaluations with respect to test functions of higher-order. This technique is designed for quadrilateral (or hexahedral) triangulations and gives local error indicators in terms of nodal contributions. We show reliability and efficiency of the estimator. Moreover, we present a simplification which is attractive from computational point of view as well.

Key words. error estimates, adaptivity, finite elements

1. Introduction

The use of locally refined meshes for the numerical solution of partial differential equations may lead to efficient numerical methods. In adaptive algorithms, an important issue is the a posteriori error estimation and the extraction of local error indicators in order to decide which cells have to be refined.

The technique of a posteriori error estimation for finite element discretizations goes back to Babuška and Rheinboldt [3]. Since then, several alternative approaches have been proposed and analyzed, e.g., residual based indicators [1], and hierarchical estimators [4, 12]. Moreover, we like to refer the reader to the books of Ainsworth and Oden [2] and of Babuška and Strouboulis [10] for an overview of different techniques. An important step for a posteriori error estimation is the work of Verfürth [11] because it was not only shown that the proposed estimator is reliable but also efficient, i.e. the estimator can be bounded by the discretitzation error multiplied by a mesh size independent constant.

In this work, we propose a posteriori error estimators for bilinear finite elements which are based on the evaluation of residuals with respect to test functions of higher-order (bi-quadratic). In relation to the standard estimators of [11], we show that these estimators are locally equivalent. From the practical point of view, the estimator has the advantage that the computation of jump terms are not necessary. This is in particular advantageous on quadrilateral meshes with hanging nodes.

A second version of the estimator is even more attractive because it is cheaper in terms of numerical costs. We show the relation of this technique to established numerical techniques of dual weighted residuals (DWR). Some numerical examples illustrate the practical behaviour and show the reliability and efficiency.

The paper is structured as follows. In section 2, we formulate the Poisson problem and its discretization by finite elements. We recall the a posteriori error estimator proposed in [11]. In section 3, the hierarchical estimator is introduced and the relation to the estimator of the previous section is discussed. Moreover, we address shortly the relation to the implicit estimator of [3]. The modified and cheaper version is described in section 4. The basic idea is to use a coarser mesh for the

Received by the editors October 6, 2011 and, in revised form, April 20, 2012.
2000 *Mathematics Subject Classification.* 65N15, 65N30.
This work is partially supported by the DFG Priority Program SPP 1276 (MetStröm). This support is gratefully acknowledged.

evaluation of the residuals. The last section is devoted to some numerical examples in2D and 3D.

2. The model problem and its discretization

2.1. Variational formulation of the Poisson problem. We consider the Poisson problem with homogeneous Dirichlet boundary conditions in a two-dimensional polygonal domain $\Omega \subset \mathbb{R}^2$:

$$-\Delta u \;=\; f \text{ in } \Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega\,. \tag{1}$$

All results carry over to mixed Dirichlet-Neumann conditions with the usual modifications.

In order to formulate the variational formulation we use the standard notations: for any open subset $\omega \subset \Omega$ let $L^2(\omega)$ be the Lebesgue space of square-integrable functions over ω, and $H^k(\omega)$ the Sobolev space with weak derivatives up to order $k \in \mathbb{N}$. The corresponding norms are denoted by $\|\cdot\|_\omega$ and $\|\cdot\|_{k;\omega}$, respectively. The L^2-scalar product and norm is denoted by $(\cdot,\cdot)_\omega$ and $\|\cdot\|$, respectively. In the case $\omega = \Omega$, we simply use $\|\cdot\|$, $\|\cdot\|_k$ and $(\cdot,\cdot)$. Furthermore, the Hilbert space of H^1 functions with vanishing traces on the boundary is denoted by $V := H_0^1(\Omega) = \{\phi \in H^1(\Omega) : \phi = 0 \text{ a.e. on } \partial\Omega\}$.

In the variational formulation, we seek for given right hand side $f \in L^2(\Omega)$ the function $u \in V$ such that

$$(\nabla u, \nabla\phi) \;=\; (f,\phi) \qquad \forall \phi \in V\,.$$

Due to the Theorem of Riesz, there is always a unique solution.

2.2. Discretization with finite elements. Let $\{\mathcal{T}_h\}_{h>0}$ be a shape regular family of triangulations of Ω consisting of triangles or quadrilaterals (but not both at the same time). For given h and $T \in \mathcal{T}_h$, h_T and ρ_T denote the diameter and the inner radius of T, respectively. The set of internal edges of $\mathcal{T}_h$ will be denoted by $\mathcal{E}_h$, i.e. for each edge $e \in \mathcal{E}_h$ the intersection $e \cap \partial\Omega$ does contain at most two boundary points. The shape regularity implies that the diameters $h_T, h_{T'}$ of two neighbouring cells $T, T' \in \mathcal{T}_h$ and the length h_e of a neighbouring edge e scale similar up to a h-independent constant:

$$0 \;<\; \max\{h_T, h_{T'}, h_e\} \;\;\leq\;\; c\min\{h_T, h_{T'}, h_e\}\,.$$

For triangular meshes, we use the space of polynomials up to degree r, denoted by $\mathcal{P}_r$. For quadrilateral meshes, the space of polynomials up to total degree r, denoted by $\mathcal{Q}_r$, is used. The finite element space is

$$\begin{aligned} \text{for tri's:} \quad V_h^{(r)} \;&:=\; \{\phi \in V : \phi|_T \in \mathcal{P}_r\,, \forall T \subset \mathcal{T}_h\}\,,\\ \text{for quad's:} \quad V_h^{(r)} \;&:=\; \{\phi \in V : \phi|_T \in \mathcal{Q}_r\,, \forall T \subset \mathcal{T}_h\}\,. \end{aligned}$$

The space of (bi-)linear elements is simply denoted by $V_h := V_h^{(1)}$. The space of piece-wise constant function is denoted by $V_h^{(0)}$.

With these notations, the corresponding finite element formulation reads

$$u_h \in V_h : \qquad (\nabla u_h, \nabla\phi) = (f,\phi) \quad \forall \phi \in V_h\,. \tag{2}$$