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FINITE ELEMENT METHODS FOR OPTIMAL CONTROL PROBLEMS GOVERNED BY LINEAR QUASI-PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, the mathematical formulation for a quadratic optimal control problem governed by a linear quasi-parabolic integro-differential equation is studied, the optimality conditions are derived, and then the a priori error estimate for its finite element approximation is given. Furthermore some numerical tests are performed to verify the theoretical results.

Key words. optimal control, linear quasi-parabolic integro-differential equations, optimality conditions, finite element methods, a priori error estimate.

1. Introduction

Linear quasi-parabolic integro-differential equations and their control appear in many scientific problems and engineering applications such as biology mechanics, nuclear reaction dynamics, heat conduction in materials with memory, and visco-elasticity, etc.. The existence and uniqueness of the solution of the linear quasi-parabolic integro-differential equations have been studied by Wheeler M. F. in [17]. Furthermore the finite element methods for linear quasi-parabolic integro-differential equations with a smooth kernel have been discussed in, e.g., X. Cui [2]. However there exists little research on optimal control problems governed by quasi-parabolic integro-differential equations, in spite of the fact that such control problems are often encountered in practical engineering applications and scientific computations. Furthermore the finite element methods of the optimal control problem governed by such equations have not been studied although there has existed much research on the finite element approximations of quasi-parabolic integro-differential equations.

Finite element approximations of optimal control problems governed by various partial differential equations have been extensively studied in the literature. There have been extensive studies in convergence of the standard finite element approximation of optimal control problems, for examples, see [1, 4, 5, 10, 11, 12, 13, 14, 15, 16]. For optimal control problems governed by linear PDEs, the optimality conditions and their finite element approximation and the a prior error estimates were established long ago, for example, see [4, 7]. The purpose of this paper is to study the mathematical formulation and its finite element approximation of the optimal control problem governed by a linear quasi-parabolic integro-differential equation. In particular we establish the optimality conditions and analyze the a priori error estimates for these constrained optimal control problems. We also present some numerical tests to verify the theoretical analysis.

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The outline of the paper is as follows. In Section 2, we present the weak formulation and analyze the existence of the solution for the optimal control problem. In Section 3, we give the optimality conditions and the finite element approximation of the optimal control problems. In Section 4, we derive the a priori error estimates for the finite element approximation of the control problem. In the last section, we perform some numerical tests, which illustrate the theoretical results.

2. Model problem and its weak formulation

Let Ω, with Lipschitz boundary $\partial\Omega$, and Ω_U be bounded open sets in $\mathbb{R}^d$, $1 \leq d \leq 3$, and $T > 0$. Introduce the objective functional

$$J(u,y) = \Big\{\frac{1}{2}\int_0^T \int_\Omega |y - z_d|^2 + \frac{\alpha}{2}\int_0^T \int_{\Omega_U} |u|^2\Big\},$$

where α is a positive regularity constant. We investigate the optimal control problem governed by a quasi-parabolic integro-differential equation as follows:

$$\min_{u \in U_{ad}} J(u, y(u)) \tag{1}$$

subject to

$$\begin{cases} y_t - \nabla\cdot\big(A\nabla y_t + D\nabla y + \displaystyle\int_0^t C(t,\tau)\nabla y(\tau)d\tau\big) & \\ = f + Bu & \text{in } \Omega\times(0,T], \\ y = 0 & \text{on } \partial\Omega\times[0,T], \\ y|_{t=0} = y^0 & \text{in } \Omega, \end{cases} \tag{2}$$

where u is the control, y is the state, z_d is the observation, U_{ad} is a closed convex subset with respect to the control, f, z_d and y^0 are some given functions to be specified later, and

$$A = (a_{ij}(\mathbf{x}))_{d\times d},\ \ D = (d_{ij}(\mathbf{x}))_{d\times d},\ \ C = (c_{ij}(\mathbf{x},t,\tau))_{d\times d},$$

B is a bounded operator independent of t from $L^2(0,T;L^2(\Omega_U))$ to $L^2(0,T;L^2(\Omega))$.

We give the weak formulation of the problem mentioned-above and study the existence and regularity of the solution. To this end, let us introduce some Sobolev spaces. Throughout the paper, we adopt the standard notations, such as $W^{m,s}(\Omega)$, for Sobolev spaces on Ω with norm $\|\cdot\|_{m,s,\Omega}$ and semi-norm $|\cdot|_{m,s,\Omega}$ for $m \geq 0$ and $1 \leq s \leq \infty$. Set $W_0^{m,s}(\Omega) = \big\{w \in W^{m,s}(\Omega) : w|_{\partial\Omega} = 0\big\}$. Also denote $W^{m,2}(\Omega)$ ($W_0^{m,2}(\Omega)$) by $H^m(\Omega)$ ($H_0^m(\Omega)$), with norm $\|\cdot\|_{m,\Omega}$, and semi-norm $|\cdot|_{m,\Omega}$. Denote by $L^r(0,T;W^{m,s}(\Omega))$ the Banach space of all L^r integrable functions from $(0,T)$ into $W^{m,s}(\Omega)$ with norm $\| v \|_{L^r(0,T;W^{m,s}(\Omega))} = \big(\int_0^T \| v \|^r_{W^{m,s}(\Omega)}\, dt\big)^{\frac{1}{r}}$ for $1 \leq r \leq \infty$. Similarly, one can define the spaces $H^1(0,T;W^{m,s}(\Omega))$ and $C^k(0,T;W^{m,s}(\Omega))$. The details can be found in [8]. To fix idea, we shall take the state space $W = L^2(0,T;V)$ with V $= H_0^1(\Omega)$ and the control space $X = L^2(0,T;U)$ with $U = L^2(\Omega_U)$. Let the observation space $Y = L^2(0,T;H)$ with $H = L^2(\Omega)$ and $U_{ad} \subseteq X$ a convex subset. In addition c or C denotes a general positive constant independent of unknowns and the meshes parameters introduced later. Introduce L^2-inner products:

$$(f_1, f_2) = \int_\Omega f_1 f_2 \ \ \forall\ f_1,\ f_2 \in H, \ \ (u,v)_U = \int_{\Omega_U} uv \ \ \forall\ u,\ v \in U$$