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HYBRID STRESS FINITE VOLUME METHOD FOR LINEAR ELASTICITY PROBLEMS

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Abstract. A hybrid stress finite volume method is proposed for linear elasticity equations. In this new method, a finite volume formulation is used for the equilibrium equation, and a hybrid stress quadrilateral finite element discretization, with continuous piecewise isoparametric bilinear displacement interpolation and two types of stress approximation modes, is used for the constitutive equation. The method is shown to be free from Poisson-locking and of first order convergence. Numerical experiments confirm the theoretical results.

Key words. finite volume method, hybrid stress method, quadrilateral element, Poisson-locking.

1. Introduction

The Finite Volume Method (FVM) is a popular class of discretization techniques for partial differential equations. One main reason for its increasing popularity is that FVM combines the geometric flexibility of the Finite Element Method (FEM) with the local conservation of physical quantities; see [27] for more interesting properties of FVM. By these virtues, FVM has been extensively used in the fields of Computational Fluid Dynamics (CFD), heat and mass transfer (see, eg [20, 22, 25, 30, 41, 44]).

In the context of Computational Solid Mechanics (CSM), however, the use of FVM has not been further explored, whereas FEM plays the dominate role because of its runaway success. Recently to simulate of multiphysical problems using flow, solid mechanics, electromagnetic, heat transfer, etc. in a coupled manner, there is increasing demand to discretize the solid mechanics using FVM [17].

Wilkins [47] made an early attempt of using FVM concept in CSM by using an alternative approximation to derivatives in a cell. Oñate, Cervera and Zienkiewicz [29] showed that FVM could be considered to be a particular case of FEM with a non-Galerkin weighting. In recent years, there has been much effort in the development and numerical investigation of FVM in CSM (see, eg [6, 9, 18, 19, 24, 40, 46]).

In this paper, we shall construct a coupling method of FVM and the hybrid stress FEM [32, 36, 50] for linear elasticity problems and present a complete numerical analysis for *a priori* error estimates. The idea follows from Wapperom and Webster [45], where FEM and FVM was coupled to simulate viscoelastic flows, and from Chen [12], where a class of high order finite volume methods was developed for second order elliptic equations by combining high order finite element methods and a linear finite volume method. We use hybrid stress FEM for the constitutive equation, and FVM for the equilibrium equation by introducing piecewise constant test functions in a dual mesh. We choose PS-stress mode [32] or ECQ4-mode [50] to approximate the stress tensor, and use isoparametric bilinear element to approximate the displacement. By doing so, our new method can inherit some virtues

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of hybrid stress FEM, e.g. the robustness with respect to Poisson locking. Meanwhile, the equilibrium equation holds locally (on every control volume). Note that the governing equations for solid body and fluid mechanics are the same but only differ in constitutive relations. Our method can be readily used to simulate the coupling of fluid flows and solid body deformation.

We shall analyze our new method following the mixed FEM theory [8, 10]. To the authors best knowledge, there are only handful rigorous analysis of mixed FVM on general quadrilateral meshes for elliptic equations [13, 14, 15, 16] and no such results for linear elasticity. Our discretization will result in a generalized saddle point system in the form

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}^T \\ \mathcal{C} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma} \\ \boldsymbol{u} \end{pmatrix} = \begin{pmatrix} 0 \\ \boldsymbol{f} \end{pmatrix}. \tag{1}$$

The analysis of the saddle point system (1) is much more involved than the symmetric case $\mathcal{B} = \mathcal{C}$. In addition to the verification of the inf-sup condition for operators $\mathcal{B}$ and $\mathcal{C}$, we need to prove that their kernels match: $\dim(\ker(\mathcal{B})) = \dim(\ker(\mathcal{C}))$ and a inf-sup condition of $\mathcal{A}$ on these two null spaces [8, 10]. Fast solvers for the non-symmetric saddle point system (1) is also more difficult than the symmetric case.

We shall overcome these difficulties by a perturbation of $\mathcal{B}$ to $\tilde{\mathcal{B}}$ using the technique developed in [54]. We show that $\tilde{\mathcal{B}} = \mathcal{D}\mathcal{C}$ with a symmetric and positive definite matrix $\mathcal{D}$. Therefore $\ker(\tilde{\mathcal{B}}) = \ker(\mathcal{C})$ and furthermore, by a scaling, (1) becomes symmetric. Note that although our system is in the mixed form, the stress unknowns can be eliminated element-wise and the resulting Schur complement is symmetric and positive definite (SPD). We can then solve this SPD system efficiently by using multigrid solvers or preconditioned conjugate gradient method with multilevel preconditioners.

In this paper, we use notation $a \lesssim b$ (or $a \gtrsim b$) to represent that there exists a constant C independent of mesh size h and the Lamé constant λ such that $a \leq Cb$ (or $a \geq Cb$), and use $a \eqsim b$ to denote $a \lesssim b \lesssim a$.

The rest of this paper is organized as follows. In section 2, we describe the model problem, introduce the isoparametric bilinear element, and review the hybrid stress FEM. Section 3 defines our hybrid stress finite volume method based on PS or ECQ4 stress mode. Section 4 presents stability analysis. Section 5 derives *a priori* error estimates. In the final section, we give some numerical results in support of theoretical ones.

2. Preliminary

In this section, we present the model problem and introduce isoparametric elements and hybrid finite element methods.

2.1. A model problem. Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary $\Gamma = \Gamma_D \cup \Gamma_N$, where $\text{meas}(\Gamma_D) > 0$. We consider the following linear elasticity problem

$$\begin{cases} -\text{div}\ \boldsymbol{\sigma} = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{\sigma} = \mathbb{C}\epsilon(\boldsymbol{u}) & \text{in } \Omega, \\ \boldsymbol{u} = 0 & \text{on } \Gamma_D, \\ \boldsymbol{\sigma n} = \boldsymbol{g}, & \text{on } \Gamma_N \end{cases} \tag{2}$$

where $\boldsymbol{\sigma} \in \mathbb{R}^{2\times 2}_{\text{sym}}$ denotes the symmetric stress tensor field, $\boldsymbol{u} = (u, v)^T \in \mathbb{R}^2$ the displacement field, $\epsilon(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$ the strain tensor, $\boldsymbol{f} \in \mathbb{R}^2$ the body