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MIXED FOURIER-GENERALIZED JACOBI RATIONAL SPECTRAL METHOD FOR TWO-DIMENSIONAL EXTERIOR PROBLEMS

JINGXIA WU AND ZHONGQING WANG*

Abstract. In this paper, we develop a mixed Fourier-generalized Jacobi rational spectral method for two-dimensional exterior problems. Some basic results on the mixed Fourier-generalized Jacobi rational orthogonal approximations are established. Two model problems are considered. The convergence for the linear problem is proved. Numerical results demonstrate its spectral accuracy and efficiency.

Key words. Mixed Fourier-generalized Jacobi rational orthogonal approximations, spectral method, exterior problems.

1. Introduction

In the past several decades, spectral method has become increasingly popular in scientific computing and engineering applications (cf. [2, 5, 6, 7, 11, 20] and the references therein). Recently, more and more attentions were paid to its applications to numerical solutions of exterior problems (cf. [9, 10, 16, 17, 22, 25, 26, 28, 29]). Most existing literature of spectral method concerning exterior problems is based on Laguerre polynomial/function approximations. For instance, Guo, Shen and Xu [16] and Zhang and Guo [29] developed the mixed spectral methods for two-/three-dimensional exterior problems, by taking Laguerre polynomials as the basis functions. While Zhang, Wang and Guo [28] and Wang, Guo and Zhang [26] studied the mixed spectral methods for two-/three-dimensional exterior problems, by taking Laguerre functions as the basis functions. Besides, some authors also considered the pseudospectral method for symmetric solutions of certain specific exterior problems, which are reduced to one-dimensional problems on the half line, see [17, 25].

On the other hand, spectral methods based on rational approximations are developed rapidly, which are also very effective for simulating numerically various partial differential equations (PDEs) on unbounded domains, see [3, 4, 8, 14, 15]. By using this approach, we could also approximate differential equations on unbounded domains directly, without any artificial boundary and variable transformation. However, the existing rational functions are usually induced by the Legendre or Chebyshev polynomials. Accordingly, the weight functions of the corresponding orthogonal systems are fixed, which might not be the most appropriate in many cases. This drawback limits the applications of rational spectral method seriously.

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A natural idea is to construct an orthogonal system of rational functions induced by the Jacobi polynomials, so that the related rational spectral method is available for more practical problems, see [24]. But the orthogonal system given in [24] is induced by the standard Jacobi polynomials. Hence its application is still limited. Recently, some authors introduced a family of generalized Jacobi orthogonal polynomials/functions, see [12, 13, 21]. Meanwhile, Guo and Yi [18], and Yi and Guo [27] investigated the generalized Jacobi rational orthogonal approximations on unbounded domains, which enlarges applications of rational spectral method. The previous statements motivate us further study and applications of the generalized Jacobi rational spectral method for exterior problems.

This paper is devoted to the mixed Fourier-generalized Jacobi rational spectral method for two-dimensional exterior problems. We shall establish some basic results on the mixed Fourier-generalized Jacobi rational orthogonal approximations. As examples, we design the mixed spectral schemes for two model problems and analyze the numerical error of the linear problem. Especially, taking suitable base functions, the resultant linear discrete systems are symmetric and sparse. Thereby, we can resolve them efficiently. The suggested method also provides accurate numerical solutions with the spectral accuracy.

The paper is organized as follows. In Section 2, we establish some basic results on the mixed Fourier-generalized Jacobi rational orthogonal approximations. In Section 3, we propose the mixed spectral method for a linear model problem and analyze its numerical error. In Section 4, we present some numerical results for two model problems. The final section is for some concluding remarks.

2. Mixed orthogonal approximations

In this section, we derive some results on the mixed Fourier-generalized Jacobi rational orthogonal approximations.

2.1. Generalized Jacobi rational orthogonal approximations. Let $\omega^{\alpha,\beta}(y) = (1-y)^{\alpha}(1+y)^{\beta}$. Denote the standard Jacobi polynomials by $J_n^{\alpha,\beta}(y),\ \alpha,\ \beta > -1,\ n \geq 0$. Let $\Gamma(y)$ be the Gamma function. For $\alpha,\ \beta > -1$, the set of Jacobi polynomials forms a complete $L^2_{\omega^{\alpha,\beta}}(-1,1)$-orthogonal system, i.e.,

$$\int_{-1}^{1} J_m^{\alpha,\beta}(y) J_n^{\alpha,\beta}(y)\omega^{\alpha,\beta}(y)dy = \gamma_n^{\alpha,\beta}\delta_{m,n}, \tag{2.1}$$

where $\delta_{m,n}$ is the Kronecker symbol, and

$$\gamma_n^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}.$$

The Jacobi polynomials fulfill the recurrence relation (cf. [23]):

$$\begin{aligned} &2n(n+\alpha+\beta)(2n+\alpha+\beta-2)J_n^{\alpha,\beta}(y) \\ &= (2n+\alpha+\beta-1)\big[(2n+\alpha+\beta)(2n+\alpha+\beta-2)y+\alpha^2-\beta^2\big]J_{n-1}^{\alpha,\beta}(y) \\ &-2(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)J_{n-2}^{\alpha,\beta}(y). \end{aligned} \tag{2.2}$$

For convenience of statements, we denote the set of real numbers by $\mathbb{R}$, the set of positive integers by $\mathbb{N}$, and the set of negative integers by $\mathbb{N}^-$. For any $\alpha, \beta \in \mathbb{R}$,