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CONVERGENCE OF DISCONTINUOUS TIME-STEPPING SCHEMES FOR A ROBIN BOUNDARY CONTROL PROBLEM UNDER MINIMAL REGULARITY ASSUMPTIONS

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Abstract. The minimization of the energy functional having states constrained to semi-linear parabolic PDEs is considered. The controls act on the boundary and are of Robin type. The discrete schemes under consideration are discontinuous in time but conforming in space. Stability estimates are presented at the energy norm and at arbitrary times for the state, and adjoint variables. The estimates are derived under minimal regularity assumptions and are applicable for higher order elements. Using these estimates and an appropriate compactness argument (see Walkington [49, Theorem 3.1]) for discontinuous Galerkin schemes, convergence of the discrete solution to the continuous solution is established. In addition, a discrete optimality system is derived and convergence of the corresponding discrete solutions is also demonstrated.

Key words. Discontinuous Time-Stepping Schemes, Finite Element Approximations, Robin Boundary Control, Semi-linear Parabolic PDEs.

1. Introduction

Space-time approximations of an optimal Robin boundary control problem are examined by using discontinuous time-stepping Galerkin schemes. In particular, the optimal control problem considered here is associated to the minimization of the energy functional,

$$J(y,g) = \frac{1}{2}\int_0^T \|\nabla y\|^2_{L^2(\Omega)}dt + \frac{\alpha}{2}\int_0^T \|g\|^2_{L^2(\Gamma)}dt \tag{1.1}$$

subject to the constraints,

$$\begin{cases} y_t - \eta\Delta y + \phi(y) &= f & \text{in } (0,T)\times\Omega \\ y + \lambda^{-1}\eta\dfrac{\partial y}{\partial \mathbf{n}} &= g & \text{on } (0,T)\times\Gamma \\ y(0,x) &= y_0 & \text{in } \Omega. \end{cases} \tag{1.2}$$

Here, Ω denotes a bounded domain in $\mathbb{R}^2$, with Lipschitz boundary Γ, λ, η positive constants, y_0, f denote the initial data and the forcing term respectively, satisfying minimal regularity assumptions, i.e.,

$$y_0 \in L^2(\Omega), \qquad f \in L^2[0,T;H^1(\Omega)^*].$$

The boundary control g is of Robin type, and α is a penalty parameter. The nonlinear mapping ϕ is monotone and continuous satisfying certain growth conditions. Several results regarding the analysis of optimal boundary control problems can be found in [24, 34, 41, 48] (see also references within). In this work we are interested in analyzing discontinuous time-stepping schemes of arbitrary order, for the gradient minimization problem with boundary controls of Robin type. There are

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several structural difficulties involved in the analysis of numerical schemes of such boundary control problems.

- The minimization of the energy functional combined with the rough initial data and forcing term, severely restricts the regularity of the state y and the control g. Therefore, standard numerical techniques employed for uncontrolled parabolic equations typically fail, since they demand more regularity of y_t than anticipated.
- In addition, the associated first order necessary conditions consist of a (backward in time) adjoint equation which is coupled to the primal (forward in time) equation through an optimality condition on the boundary, and nonlinear terms (see, e.g [24, 34, 41, 48]). This leads to reduced regularity for the state and adjoint variables, and hence numerical analysis approaches based on standard 'boot-strap' techniques are not directly applicable, in particular in presence of $L^2(\Omega)$ initial data.
- If higher norms of the control g are included in the functional then a boot-strap argument can be applied to recover some additional regularity on the adjoint variable, taking into account the linear structure of the adjoint PDE, and the zero terminal data. However this approach typically leads to an optimality condition of a PDE form on the boundary which is hard to solve computationally.
- Due to the lack of regularity, the recovery of the necessary compactness of discrete schemes, via discrete version of the classical Aubin-Lions Lemma for nonlinear PDEs (see e.g. [44, 51]) is not evident.
- The parameter α effectively determines the "size" of the control g, which is needed in order to minimize the gradient. As a consequence the dependence of various stability constants upon α should be tracked. For some relevant discussions for the velocity tracking problem we refer the reader to [24].

To overcome these difficulties we analyze a classical discontinuous Galerkin scheme which is discontinuous in time and conforming in space. It is well known that discontinuous time-stepping schemes perform well for problems which satisfy low regularity properties. As we will subsequently show for the discrete control problem, the discontinuous time-stepping schemes inherit crucial regularity and stability properties of the continuous weak formulation of the underlying PDE, such as estimates under minimal regularity assumptions at arbitrary time points (see e.g. [10, 11, 49] for the uncontrolled evolutionary PDEs). Such estimates allow the use of the recently developed discrete compactness property of discontinuous time-stepping schemes under minimal regularity assumptions (see Walkington, [49, Theorem 3.1]), within the optimal control setting.

As a consequence, strong convergence in an appropriate norm is established, and hence the semi-linear terms are treated by embedding theorems. Using the above technique, we prove convergence of the discrete optimal solution to the continuous problem. In addition, a "boot-strap" argument can be rigorously applied in order to derive the associated discrete optimality system (discrete first order necessary condition) and then to prove convergence of the corresponding discrete adjoint variable, without requiring additional regularity on the time-derivatives. A novel element of the proposed methodology for the boundary control problem with semi-linear state constraints is that the time discretization step length τ can be chosen independently of the spatial discretization parameter h. In addition, the dependence upon λ, α of various stability constants is carefully tracked. The emphasis