INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 10, Number 3, Pages 712–726

A SEMIDISCRETE APPROXIMATION SCHEME FOR NEUTRAL DELAY-DIFFERENTIAL EQUATIONS

R. H. FABIANO

Abstract. We consider an approximation scheme for systems of linear delay-differential equations of neutral type. The finite dimensional approximating systems are constructed with basis functions defined using linear splines, extending to neutral equations a scheme which had previously been defined only for retarded equations. A Trotter-Kato semigroup convergence result is proved, and numerical results are given to illustrate the qualitative behavior of the scheme.

Key words. neutral delay equation, semidiscrete approximation, semigroup theory

1. Introduction

In this paper we consider semidiscrete approximation schemes for linear autonomous neutral delay differential equations. In particular, the neutral equation is formulated as a linear system on an infinite dimensional Hilbert space, and this system is approximated by a sequence of linear differential equations on finite dimensional Hilbert spaces. A Trotter-Kato type theorem is used to argue convergence.

The idea of using this type of semigroup-theoretic finite dimensional approximation for delay differential equations has been known for some time. It has often been the case that an approximation scheme is developed first for retarded equations and later extended to neutral equations (this has often been the case for the development of other parts of the theory for delay differential equations as well). Perhaps the earliest paper with a rigorous implementation of this idea (that is, rigorous justification of both well-posedness as well as Trotter-Kato type semigroup convergence) is [1]. There Banks and Burns prove convergence of the so-called averaging approximation scheme (the basis functions are piecewise constant) for linear retarded delay equations, and the scheme is applied to a control problem. The averaging scheme for retarded equations was extended to neutral equations by Kappel and Kunisch in [19] and [23]. Meanwhile in [3] Banks and Kappel construct an approximation scheme for retarded delay equations which uses certain splines (in particular, splines which are restricted to be in the domain of the infinitesimal generator of the semigroup associated with the equation) as basis functions for the finite dimensional approximation spaces. They show this scheme obtains better convergence rates than the averaging scheme when applied to retarded equations. This spline based scheme was extended to neutral equations by Kappel and Kunisch in [19] and [20]. Later in [2] Banks, Ito, and Rosen observed numerically that this spline based scheme performed poorly (that is, worse than the numerically observed performance of the averaging scheme) when used to approximate feedback gains in an optimal control problem for a retarded equation. The authors in [2] conjectured that the spline based scheme of [3] did not yield convergence for the adjoint semigroup, and this conjecture was confirmed by Burns, Ito, and Propst

Received by the editors June 12, 2012.
2000 *Mathematics Subject Classification.* 34K06, 34K28, 34K40, 47D06, 65L03.

in [8]. In [21] Kappel and Salamon developed an improved spline based approximation scheme for retarded delay equations (in particular, the domain restriction on the splines is removed), and the scheme yielded both semigroup convergence and adjoint semigroup convergence. In the present paper we extend this improved spline scheme to neutral delay equations, and prove semigroup convergence. This extends results in [14] which considered the scalar, single delay neutral equation. The adjoint semigroup convergence has been verified for the scalar single delay neutral case in [10], but the extension to the general multiple delay neutral systems under consideration in this paper is a subject for further investigation. An additional contribution of this paper is a variational version of the Trotter-Kato theorem which is especially suitable for our construction. This paper is concerned only with semidiscrete approximation of linear neutral delay equations. For fully discrete and other approaches to approximation of delay equations, we refer to [4], [5], [26], and the references therein.

We now define the system of linear neutral delay-differential equations under consideration. Given $\eta_0 \in \mathbb{C}^n$ and $\phi_0 \in L^2(-r, 0; \mathbb{C}^n)$, consider the initial value problem

$$
\begin{aligned}
\frac{d}{dt}[x(t) + \sum_{k=1}^{m} C_k x(t - r_k)] \quad &= \quad Ax(t) + \sum_{k=1}^{m} B_k x(t - r_k), \\
x(0) + \sum_{k=1}^{m} C_k x(-r_k) = \eta_0, \quad x(\theta) \quad &= \quad \phi_0(\theta), \quad -r \le \theta < 0,
\end{aligned} \tag{1}
$$

where B_k, C_k, $k = 1, \dots, m$ and A are $n \times n$ matrices with complex entries, and $0 = r_0 < r_1 < \cdots < r_m \equiv r$. In a standard fashion first described in [7], the initial value problem (1) can be reformulated as an abstract Cauchy problem on a Hilbert space. In particular, define the Hilbert space $X = \mathbb{C}^n \times L^2(-r, 0; \mathbb{C}^n)$ endowed with the norm

$$
\|(\eta, \phi)\|_X^2 = \|\eta\|^2 + \int_{-r}^{0} \overline{\phi(\theta)}^T G(\theta)\phi(\theta)\, d\theta. \tag{2}
$$

Here and throughout the paper we use the unsubscripted norm notation $\|\cdot\|$ to denote the standard Euclidean norm on $\mathbb{C}^n$ or its induced matrix norm on $n \times n$ matrices. It is clear that the norm $\|\cdot\|_X$ depends on the choice of the $n \times n$ matrix-valued weight function G, although we do not indicate this explicitly in the notation. However we take as a standing assumption that the weight function G is chosen so the norm $\|\cdot\|_X$ is equivalent to the usual energy norm which corresponds to $G(\theta) \equiv I$. (Later we shall impose further restrictions on the function G so as to obtain an important dissipative inequality). We may also write the norm as

$$
\|(\eta, \phi)\|_X^2 = \|\eta\|^2 + \sum_{k=1}^{m} \int_{-r_k}^{-r_{k-1}} \overline{\phi(\theta)}^T G(\theta)\phi(\theta)\, d\theta
$$

with compatible inner product

$$
\langle (\eta, \phi), (\xi, \psi) \rangle_X = \overline{\xi}^T \eta + \sum_{k=1}^{m} \int_{-r_k}^{-r_{k-1}} \overline{\psi(\theta)}^T G(\theta)\phi(\theta)\, d\theta. \tag{3}
$$

Next define the linear operator $\mathcal{A} : \operatorname{dom} \mathcal{A} \subset X \to X$ on the domain

$$
\operatorname{dom} \mathcal{A} = \{(\eta, \phi) \in X : \phi \in H^1(-r, 0; \mathbb{C}^n),\ \eta = \phi(0) + \sum_{k=1}^{m} C_k \phi(-r_k)\},
$$