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THE FINITE ELEMENT METHOD OF A EULER SCHEME FOR STOCHASTIC NAVIER-STOKES EQUATIONS INVOLVING THE TURBULENT COMPONENT

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Abstract. In this paper we study the finite element approximation for stochastic Navier-Stokes equations including a turbulent part. The discretization for space is derived by finite element method, and we use the backward Euler scheme in time discretization. We apply the generalized L_2-projection operator to approximate the noise term. Under suitable assumptions, strong convergence error estimations with respect to the fully discrete scheme are well proved.

Key words. stochastic Navier-Stokes equations, finite element method, discrete scheme, and error estimation.

1. Introduction

Let $\Omega \in R^2$ be a bounded convex polygonal domain with boundary $\partial\Omega$. In this paper, we consider finite element approximation of stochastic Navier-Stokes equations with the turbulent term

$$\partial_t u = \triangle u - (u \cdot \nabla)u - \nabla p + f(u) + [(\sigma \cdot \nabla)u - \nabla \tilde{p} + g(u)]\dot{W}, \tag{1}$$

$$u(0) = u_0, \quad \nabla \cdot u = 0, \tag{2}$$

on Ω in a finite time interval $[0, T]$. The turbulent term is driven by the white noise $\dot{W}$. In this article, $\dot{W}$ denotes a time derivative of a Hilbert space valued Wiener process. Assumptions on other functions will be specified later.

The stochastic Navier-Stokes equation, which displays the behavior of a viscous velocity field of an incompressible liquid, is widely regarded as one of the most fascinating problems of fluid mechanics, see [1]. A. Bensoussan and R. Temam generally analyze Navier-Stokes equations driven by white noise type random force in [2]. Later, the existence, the uniqueness and other properties of the generalized solutions with respect to stochastic equations have been extensively researched by many authors; see [4], [9], [15], [18], etc. An overview of some developments in the ergodic theory of the stochastically forced Navier-Stokes equations are presented by Jonathan C. Mattingly in [6].

The effective researches about unsteady incompressible stochastic Navier-Stokes equations driven by white noise are considered by R. Mikulevicius and B. L. Rozovskii; see [13] and [14] with a review of relevant recent work. Under some basic assumptions, the existence of a global weak (martingale) solution of the unsteady incompressible stochastic Navier-Stokes equation (1.1) with Cauchy problem is well proved in [17]. Furthermore, R. Mikulevicius and B. L. Rozovskii consider the corresponding fluid dynamics modeled by a stochastic flow in [16].

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The finite element method, which is a common technique for partial differential equations, is widely used to obtain finite dimensional approximations. The ideas based on finite element approximation to investigate stochastic differential equations are well studied in many literatures; see, [3], [10], [12], [19], [21], [22], [23] for some previous work. Yubin Yan consider the semidiscrete Galerkin approximation of a stochastic parabolic partial differential equation in [25]. Later, the fully finite element method for stochastic parabolic partial differential equations driven by white noise is proved and optimal strong convergence error estimates are given in [24]. Semidiscrete finite element approximation of the linear stochastic wave equation with additive noise is well studies in [11]. However, numerical analysis of unsteady incompressible stochastic Navier-Stokes equations has not been thoroughly considered. The major purpose of our paper is to study finite element approximation for unsteady incompressible stochastic Navier-Stokes equations involving the complex turbulent component. In our paper, stochastic Navier-Stokes equations are taken in the generalized sense.

The plan of this paper is as follows. In section 2 useful notations and related properties are introduced. Some important preliminaries are given. The regularity in time of the solution is deduced. In section 3 we consider finite element approximation of stochastic Navier-Stokes equations with turbulent term. The semidiscrete form and the fully discretization are obtained. In section 4 we deduce the main error estimations with respect to the fully discretization of the stochastic equations. Using above-mentioned techniques, we finally complete the proofs of strong convergence error estimates. Section 5 are our conclusions of this paper.

2. Notations and preliminaries

In this section we will introduce some useful notations and some important preliminaries.

Let H be the Hilbert space of real vector functions in $L_2(\Omega)$ with the inner product $(\cdot,\cdot)$. Given integer $m \geq 0$ and $1 \leq p < \infty$, define

$$W^{m,p}(\Omega) = \{u \in L_p(\Omega) : D^\alpha u \in L_p(\Omega), \forall \alpha, 0 \leq |\alpha| \leq m\},$$

equipped with the norm

$$||u||_{W^{m,p}(\Omega)} = ||u||_{m,p} = \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_p^p \right)^{1/p}, \ 1 \leq p < \infty.$$

Set

$$W_0^{m,p}(\Omega) = \{u \in W^{m,p}(\Omega) : u|_{\partial\Omega} = 0\}.$$

Obviously $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ stand for Sobolev spaces on Ω. More explicitly we can write $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and denote $W_0^{m,2}(\Omega)$ as $H_0^m(\Omega)$. It is easy to verify that $W^{0,p}(\Omega) = L_p(\Omega)$.

Moreover we can come to the conclusion that $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are Banach spaces, and $H^m(\Omega)$ and $H_0^m(\Omega)$ are Hilbert spaces. The relevant inner conduct is

$$(u, v)_m = \sum_{0 \leq |\alpha| \leq m} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}, \ u, v \in H^m(\Omega).$$

Obviously, the above-mentioned spaces can be extended to vector functions.

As usual, $(\Omega, \mathcal{F}, \mathbf{P})$ denotes a normal filtered probability space with a normal right continuous filtration $(\mathcal{F}_t)$. In our paper, W is a cylindrical Wiener process