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AN EFFECTIVE GRADIENT PROJECTION METHOD FOR STOCHASTIC OPTIMAL CONTROL

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Abstract. In this work, we propose a simple yet effective gradient projection algorithm for a class of stochastic optimal control problems. The basic iteration block is to compute gradient projection of the objective functional by solving the state and co-state equations via some Euler methods and by using the Monte Carlo simulations. Convergence properties are discussed and extensive numerical tests are carried out. Possibility of extending this algorithm to more general stochastic optimal control is also discussed.

Key words. stochastic optimal control, numerical method, gradient projection algorithm

1. Introduction

Stochastic optimal control is an essential tool for developing and analyzing models that have stochastic dynamics, and it has been fully developed both theoretically and practically in mathematics, physics and engineering. There has existed a very extensive body of literature in this area, and it is impossible to present an even very brief review on its development here. Some introductive accounts (more from mathematical points of view) can be found, for example, in [5, 6, 18, 30], and [11, 14]. Some of the research relevant to our work can be found in [5, 15, 18, 29], and [4, 10, 16, 31, 41, 42, 43]. Practical examples of stochastic optimal control include engineering systems [10, 27, 31, 43, 45], option pricing and portfolio optimization models from finance [25, 26, 37, 47, 50], analysis of climate change policies [1], and biological and medical applications [17].

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which is generated by a one-dimensional standard Brownian motion $\{W_t\}_{t\geq 0}$. Let $T > 0$ be a fixed real number that is called time horizon. We denote by $L^2(\Omega, \mathcal{F}_T; R)$ the space of real-valued square-integrable $\mathcal{F}_T$-measurable random variables, and by $L^2_{\mathcal{F}}([0,T]; R)$ the space of real-valued square-integrable $\mathcal{F}_t$-adapted processes such that

$$E\Big\{ \int_0^T |y_t|^2 dt \Big\} < +\infty. \tag{1}$$

In this paper we consider numerical solutions to the following stochastic control problem. The objective functional

$$J(y,u) = \int_0^T E[h(y)]dt + \int_0^T j(u)dt, \tag{2}$$

where h and j are smooth functions with the continuous first order derivatives, $u \in U_{ad}$ is a deterministic control, where U_{ad} is a close convex set in the control space $L^2(0,T)$.

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An admissible control u^* is called optimal if it attains the minimum of $J(y(u), u)$, where the state $y(u) \in L^2_{\mathcal{F}}([0,T]; R)$ is a stochastic process which is generated by

$$dy = f(t, y, u)dt + g(t, y)dW_t, \quad y(0) = y_0. \tag{3}$$

In this paper we assume that f, g are continuously differentiable with respect to (t, y, u) and (t, y), respectively, and that their derivatives are bounded.

Under the above assumptions, we know equation (3) admits a unique solution $y(\cdot) \in L^2_{\mathcal{F}}([0,T]; R)$ for the given $(y_0, u(\cdot)) \in R \times U_{ad}$ (see [22]). We call such a $y(\cdot)$ the corresponding trajectory. Let us note that here the control does not appear in the diffusion term for easy of exposition. For the general case, one would need to have more theoretical preparations on backward stochastic differential equations for a rigorous treatment of the adjoint state equations (see [6, 41, 42]), although our methods are still applicable.

In general most realistic models do not admit closed form solutions and thus effective numerical methods play a key role for practical applications of stochastic optimal control. In the literature numerous numbers of numerical methods have been proposed for stochastic optimal control and the related problems. Numerical methods used to solve stochastic optimal control have at least four broad classes: Those transferring the control problem into finite dimensional stochastic programming, see, e.g., [12, 16, 19, 20, 31, 43, 46, 48]; those based on Dynamic Programming Principle (DPP), see e.g. [7, 30], in particular those solving HJB equations for the feedback solutions - there are many references in this area, see [2, 5, 6, 13, 21] for some early work; the third class based on martingale methods, see e.g. [25, 26, 44]; and those based on the Stochastic Maximum Principle (SMP) [18]. The method proposed in this paper is based on an iterative algorithm for the solution of the SMP. There exists extensive research on the first three classes methods. Although the SMP is widely used in solving the stochastic optimal control, see, e.g., [47] and [50], it is not often used in numerical algorithms yet. The likely reasons are that it will not directly produce the feedback control as explained below, and the computation of the adjoins requires the solution of a backwards stochastic differential equation (BSDE), which is computationally expensive.

Compared with the deterministic optimal control, stochastic optimal control is much more complicated from the perspective of obtaining numerical solutions that are realizable to real applications. One of the reasons is that often the value of optimal control $u(t)$ at a time t will depend on ω (so $u(t, \omega)$) so that it is not very useful to only compute and then apply the numerical solutions of the optimal control like in the deterministic case. To be practically useful, some forms of feedback relationship between the optimal state and optimal control need to be computed numerically as well (as in the approach of Bellman Equation), as otherwise the optimal control is difficult to realize. Therefore the existing numerical methods in the literature are rather complex. In this paper we study a useful case where the control is deterministic (but the state is still stochastic) as the first step towards developing fast numerical algorithms for general stochastic optimal control. In this case the optimal control does not directly depend on ω (but depends on $y(t)$) so that it is meaningful to just compute the optimal control and apply it without the feedback laws. This is quite desirable in some business and engineering decision making where the stochastic effect is not overwhelming and thus deterministic decision rules are desirable and sufficient. A deterministic solution is also useful for future planning. Such examples can be found e.g. in [10, 31] (Engineering Control), [12] (financial) and [43] (Stochastic Hybrid Systems). In this work we are then able to derive simple yet effective numerical algorithms with convergence analysis. More