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SPATIAL ERROR ESTIMATES FOR A FINITE ELEMENT VISCOSITY-SPLITTING SCHEME FOR THE NAVIER-STOKES EQUATIONS

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Abstract. In this paper, we obtain optimal first order error estimates for a fully discrete fractional-step scheme applied to the Navier-Stokes equations. This scheme uses decomposition of the viscosity in time and finite elements (FE) in space.

In [15], optimal first order error estimates (for velocity and pressure) for the corresponding time-discrete scheme were obtained, using in particular $\mathbf{H}^2 \times H^1$ estimates for the approximations of the velocity and pressure. Now, we use this time-discrete scheme as an auxiliary problem to study a fully discrete finite element scheme, obtaining optimal first order approximation for velocity and pressure with respect to the max-norm in time and the $\mathbf{H}^1 \times L^2$-norm in space.

The proof of these error estimates are based on three main points: a) provide some new estimates for the time-discrete scheme (not proved in [15]) which must be now used, b) give a discrete version of the $\mathbf{H}^2 \times H^1$ estimates in FE spaces, using stability in the $\mathbf{W}^{1,6} \times L^6$-norm of the FE Stokes projector, and c) the use of a weight function vanishing at initial time will let to hold the error estimates without imposing global compatibility for the exact solution.

Key words. Navier-Stokes Equations, splitting in time schemes, fully discrete schemes, error estimates, mixed formulation, stable finite elements.

1. Introduction

We consider the Navier-Stokes system, modelling viscous and incompressible fluids filling a bounded domain $\Omega \subset I\!R^3$ in a time interval $(0,T)$:

$$(P) \qquad \left\{ \begin{array}{rcll} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\,\Delta\mathbf{u} + \nabla p &=& \mathbf{f} & \text{in } \Omega\times(0,T), \\ \nabla\cdot\mathbf{u} &=& 0 & \text{in } \Omega\times(0,T), \\ \mathbf{u} &=& 0 & \text{on } \partial\Omega\times(0,T), \\ \mathbf{u}_{|t=0} &=& \mathbf{u}_0 & \text{in } \Omega. \end{array} \right.$$

where $\mathbf{u} : (\mathbf{x},t) \in \Omega\times(0,T) \to I\!R^3$ the velocity field and $p : (\mathbf{x},t) \in \Omega\times(0,T) \to I\!R^3$ the pressure are the unknowns, and data are $\nu > 0$ the viscosity coefficient (which is assumed constant for simplicity) and $\mathbf{f} : (\mathbf{x},t) \in \Omega\times(0,T) \to I\!R^3$ the external forces. We denote by ∇ the gradient operator and Δ the Laplace operator.

Considering a (regular) partition of $[0,T]$ of diameter $k = T/M$: $(t_m = mk)_{m=0}^M$, for a given vector $u = (u^m)_{m=0}^M$ with $u^m \in X$ (a Banach space), let us to introduce the following notation for discrete in time norms:

$$\|u\|_{l^2(X)} = \left(k \sum_{m=0}^{M} \|u^m\|_X^2 \right)^{1/2} \quad \text{and} \quad \|u\|_{l^\infty(X)} = \max_{m=0,\dots,M} \|u^m\|_X$$

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For simplicity, we will denote $H^1 = H^1(\Omega)$ etc., $L^2(H^1) = L^2(0,T;H^1)$ etc., and $\mathbf{H^1} = H^1(\Omega)^3$ etc.

The numerical analysis for the Navier-Stokes problem (P) has received much attention in the last decades and many numerical schemes are now available. The main (numerical) difficulties in this problem are the coupling between the pressure and the incompressibility condition and the nonlinearity of the convective terms.

Fractional step methods in time are becoming widely used in this context, allowing us to separate the effects of different operators appearing in the problem. For instance, the projection schemes decompose the convection-diffusion operators to the incompresibility ([20], [21], [19], [13]). These projection schemes are two-step schemes where the second step is a free divergence projection step. The main drawbacks of projection methods are that the end-of-step velocity does not satisfy the exact boundary conditions and the discrete pressure satisfies "artificial" boundary conditions.

Another class of fractional step methods, so-called θ-schemes (where viscosity is not fully decoupled from incompressibility), were introduced by Glowinski and his co-authors in the 1980's (see for instance a review in [12]). Afterwards, some analytical results were given, see for instance [8] where stability and convergence of two fully discrete θ-schemes were proved.

In this paper, we study a fractional step method (so-called viscosity-splitting) which can be seen as an special case of the θ-scheme. This scheme was inspired in the previous projection schemes and θ-schemes, jointly to the predictor-corrector argument applied to incompressible fluids ([6]). This viscosity-splitting method was studied in [1], [2], [3] and [4]. It is a two-step scheme splitting the nonlinearity and the incompressibility of the problem into two different steps (but keeping viscosity term and boundary conditions in both steps). Essentially, in this viscosity-splitting scheme, given $\mathbf{u}_h^m$ an approximation of $\mathbf{u}(t_m)$, first one computes an intermediate velocity $\mathbf{u}_h^{m+1/2}$ (as a first approximation of $\mathbf{u}(t_{m+1})$) by means of a convection-diffusion problem, and afterwards $(\mathbf{u}_h^{m+1}, p_h^{m+1})$ (as approximation of $(\mathbf{u}(t_{m+1}), p(t_{m+1}))$) is obtained solving a generalized Stokes problem. On the other hand, the θ-scheme is a three-step method; the first and third step (or generalized Stokes problem) accounts for viscous effect together with incompressibility, but it also includes an explicit convective term; the second step (or regularized Burger's problem) also includes an implicit viscous term and a non-linear implicit approximation of convection together with an explicit pressure gradient but not the incompressibility condition.

In [1], [2], Blasco, Codina and Huerta prove the convergence of the time-discrete viscosity splitting scheme. Afterwards, also for the time-discrete case, error estimates of order $O(k)$ in $l^2(\mathbf{H}^1) \cap l^\infty(\mathbf{L}^2)$ for the end-of-step velocity $\mathbf{u}^{m+1}$ and order $O(k^{1/2})$ in $l^2(L^2)$ for the pressure p^{m+1} are obtained in [3]. Moreover, in [4] these error estimates are used to obtain the following error estimates for a fully discrete scheme based on $O(h)$ finite element approximations in $\mathbf{H}^1 \times L^2$ for the velocity and pressure:

$$\|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \le C\,(k+h),$$

under the constraint $h^2 \le C\,k$.

On the other hand, in [2] numerical computations with this viscosity-splitting scheme drive to order $O(k)$ in $L^2(\Omega)$ for velocity and pressure. In [11], this time scheme is studied jointly to Galerkin discontinuous FE methods in space with $P_1 \times P_0$ approximation. From the analytical point of view, order $O(k+h)$ in $l^\infty(\mathbf{L}^2)$ for the velocity and order $O(\sqrt{k}+h)$ in $l^2(L^2)$ for the pressure were obtained.