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ERROR ESTIMATES OF THE CRANK-NICOLSON SCHEME FOR SOLVING BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

WEIDONG ZHAO, YANG LI AND LILI JU

Abstract. In this paper, we study error estimates of a special θ-scheme – the Crank-Nicolson scheme proposed in [25] for solving the backward stochastic differential equation with a general generator, $-dy_t = f(t, y_t, z_t)dt - z_t dW_t$. We rigorously prove that under some reasonable regularity conditions on φ and f, this scheme is second-order accurate for solving both y_t and z_t when the errors are measured in the L^p $(p \geq 1)$ norm.

Key words. Backward stochastic differential equations, Crank-Nicolson scheme, θ-scheme, error estimate

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $T > 0$ a finite time, $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ a filtration satisfying the usual conditions. Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete, filtered probability space on which a standard d-dimensional Brownian motion W_t is defined and $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$. Let $L^2 = L^2_{\mathcal{F}}(0, T)$ be the set of all $\mathcal{F}_t$-adapted and mean-square-integrable vector/matrix processes. We consider the backward stochastic differential equation (BSDE)

$$-dy_t = f(t, y_t, z_t)dt - z_t dW_t, \quad \forall\, t \in [0, T), \tag{1.1}$$

with the terminal condition

$$y_T = \xi,$$

where the generator $f = f(t, y_t, z_t)$ is a vector function valued in R^m and is $\mathcal{F}_t$-adapted for each (x, y), and the terminal variable $\xi \in L^2$ is $\mathcal{F}_T$ measurable. Rewriting the BSDE (1.1) in the integral form gives us

$$y_t = \xi + \int_t^T f(s, y_s, z_s)\, ds - \int_t^T z_s\, dW_s, \quad \forall\, t \in [0, T). \tag{1.2}$$

We note that the second integral term on the right-hand side of (1.2) is an Itô-type integral. A process (y_t, z_t): $[0, T] \times \Omega \to R^m \times R^{m \times d}$ is called an L^2-solution of the BSDE (1.2) if, in the probability space $(\Omega, \mathcal{F}, ,P)$, it is $\{\mathcal{F}_t\}$-adapted, square integrable, and satisfies the integral equation (1.2) [16].

In 1990, Pardoux and Peng first proved in [16] the existence and uniqueness of the solution of general nonlinear BSDEs (i.e, f is nonlinear), and later in [17], obtained some relations between BSDEs and stochastic partial differential equations (SPDEs). Since then, the theory of BSDEs has been extensively studied by many researchers and BSDEs have found applications in many fields, such as finance, risk measure, stochastic control, and etc.. Peng obtained the relation between BSDEs and parabolic PDEs in [19], and then the generalized stochastic maximum principle

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and the dynamic programming principle for stochastic control problems based on BSDEs in [18, 21]. The nonlinear $g-$expectation via a particular nonlinear BSDE was introduced in [20], and in [7] it was found that a dynamic coherent risk measure can be represented by a properly defined g-expectation. Thus, it is very important and useful to study solutions of BSDEs.

In this paper, we consider the case of $\xi = \varphi(W_T)$, and assume that the BSDE (1.2) has a unique solution (y_t, z_t). It was shown in [19] that the solution (y_t, z_t) of (1.2) can be represented as

$$y_t = u(t, W_t), \qquad z_t = \nabla_x u(t, W_t), \qquad \forall t \in [0, T), \tag{1.3}$$

where $u(t, x)$ is the solution of the following parabolic partial differential equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} + f(t, u, \nabla_x u) = 0, \tag{1.4}$$

with the terminal condition $u(T, x) = \varphi(x)$, and $\nabla_x u$ is the gradient of u with respect to the spacial variable x. The smoothness of u clearly depends on ϕ and f.

It is well-known that it is often difficult to obtain analytic solutions of BSDEs, so that computing their approximate solutions becomes highly desired. Based on the relation between the BSDEs and the corresponding parabolic PDEs, some numerical algorithms were proposed to solve BSDEs [3, 11, 12, 13, 14, 15, 19, 24], and furthermore, a four step algorithm was proposed in [10] to solve a class of more general equations called forward-backward stochastic differential equations (FBSDEs). In [25], a family of θ-schemes were proposed for solving general BSDEs. In particular, a special case of the θ-scheme – the *Crank-Nicolson* (C-N) scheme was numerically demonstrated to be *second-order accurate*. This accuracy result was theoretically proven in [22, 26] for the simplified case that the generator function f is independent of z_t in (1.2), however, the proof for the cases of general generators remains open till now. A family of multi-step schemes were recently developed in [27] based on the Lagrange interpolation and the Gauss-Hermite quadratures. Accuracies of these multi-step schemes were numerically shown to be of high order for solving the BSDE (1.2), but again the result was only theoretically confirmed for BSDEs with a generator f independent of z_t. There are also some other numerical methods for solving BSDEs (or FBSDEs), which were proposed based on directly discretizing BSDEs or FBSDEs, see [1, 2, 4, 5, 8, 9, 21, 23, 24] and references cited therein.

The aim of this paper is to study error estimates of the special θ scheme – the Crank-Nicolson scheme for solving the general BSDE (1.2) with terminal condition $\xi = \varphi(W_T)$. For the purpose of simple representations, let us first introduce the following notations:

- $\|X\|_{L^p}$ $(p \geq 1)$: the L^p-norm for $X \in L^p$ defined by $\mathbb{E}[|X|^p]^{\frac{1}{p}}$.
- $C_b^{l,k,k}$: the set of continuously differential functions $\psi : [0, T] \times R^d \times R^{m\times d} \to R$ with uniformly bounded partial derivatives $\partial_t^{l_1}\psi$ and $\partial_y^{k_1}\partial_z^{k_2}\psi$ for $l_1 \leq l$ and $k_1 + k_2 \leq k$.
- $C_b^{l,k}$: the set of functions $\psi : (t, x) \in [0, T] \times R^d \to R$ with uniformly bounded partial derivatives $\partial_t^{l_1}\partial_x^{k_1}\psi$ for $l_1 \leq l$ and $k_1 \leq k$.
- C_b^k: the set of functions $\psi : x \in R^d \to R$ with uniformly bounded partial derivatives $\partial_x^{k_1}\psi$ for $k_1 \leq k$.
- $\mathcal{F}_s^{t,x}(t \leq s \leq T)$: the σ-field generated by the Brownian motion $\{x + W_r - W_t, t \leq r \leq s\}$ starting from the time-space point (t, x). Let $\mathcal{F}^{t,x} = \mathcal{F}_T^{t,x}$.
- $\mathbb{E}[X]$: the mathematical expectation of the random variable X.