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# AN OPTIMAL UNIFORM A PRIORI ERROR ESTIMATE FOR AN UNSTEADY SINGULARLY PERTURBED PROBLEM

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**Abstract.** A time–dependent convection–diffusion problem is discretized by the Galerkin finite element method in space with bilinear elements on a general layer adapted mesh and in time by discontinuous Galerkin method. We present optimal error estimates. The estimates hold true for consistent stabilization too.

**Key words.** discontinuous Galerkin, convection–diffusion, layer adapted mesh, error estimate

## 1. Introduction

We focus ourselves on the analysis of the solution of unsteady linear 2D singularly perturbed convection–diffusion equation. This type of equation can be considered as simplified model problem to many important problems, especially to Navier–Stokes equations.

The space discretization of such a problem is a difficult task and it stimulated development of many stabilization methods (e.g. streamline upwind Petrov–Galerkin (SUPG) method, local projection stabilization methods) and layer–adapting techniques (e.g. Shishkin meshes, Bakhvalov meshes). For the overview see [9] or [8].

In order to achieve optimal diffusion–uniform error estimates we employ layer adapted meshes. On these general layer adapted meshes we assume a general space discretization covering standard conforming finite element method (FEM) or consistent stabilization methods. The resulting system of ordinary differential equations is solved by discontinuous Galerkin (DG) method.

Considering the space discretization on Shishkin meshes, we will follow the theory for stationary singularly perturbed problems based on the solution decomposition, which enables us to derive a priori error estimates independent of the diffusion parameter even with respect to the norms (seminorms) of the exact solution, which can be also highly dependent on the diffusion parameter. For the details see [9].

The discontinuous Galerkin (DG) method is a very popular approach for solving ordinary differential equations arising from space discretization of parabolic problems, which is based on piecewise polynomial approximation in time. Among important advantages we should mention unconditional stability for arbitrary order, which allows us to solve stiff problems efficiently, and good smoothing property, which enables us to work with inexact or rough data. For introduction to DG time discretization see e.g. [11].

In [6] and [1] the authors study DG in time and DG and local projection stabilization method, respectively, in space on standard meshes for singularly perturbed problems. The error estimates in these papers contain norms of the exact solutions which go to infinity if diffusion parameter goes to zero.

There are only few papers dealing with finite elements in space on the special meshes combined with any discretization in time. While in [7] the $\theta$–scheme as discretization in time is used, in [5] the authors study BDF time discretization.

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In [7] the authors also study DG time discretization and derive suboptimal error estimates.

Our aim is improving some results from [7] and proving optimal a priori diffusion–uniform error estimates for DG time discretization in $L^\infty(L^2)$ norm.

The main difficulty in proving optimal diffusion–uniform error estimates for DG time discretization is the fact that we cannot employ standard technique of the proof, which is based on the construction of a suitable projection, which enables us to eliminate discrete time derivative in the error equation, see e.g. [10]. This technique enforces us to do some upper bound of the projection error contained in stationary terms, which depends on a higher time derivative of the exact solution in $H^1$ seminorm, which depends on the diffusion parameter.

## 2. Continuous problem

Let $\Omega = (0,1)^2$ be a computational domain and $T > 0$. Then let us consider parabolic singularly perturbed problem

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \varepsilon \Delta u + b \cdot \nabla u + cu &= f, \quad \forall x \in \Omega, t \in (0,T), \\
u &= 0, \quad \forall x \in \partial\Omega, t \in (0,T), \\
u(x,0) &= u^0(x), \quad \forall x \in \Omega,
\end{aligned} \tag{1}
$$

where function $u^0 \in L^2(\Omega)$, $0 < \varepsilon << 1$ and functions $f(x,t)$, $b(x)$ and $c(x)$ are sufficiently smooth with $b_1(x) > \beta_1 > 0$ and $b_2(x) > \beta_2 > 0$. By substitution in time variable we can achieve

$$
c - \frac{1}{2}\nabla \cdot b \geq c_0 > 0. \tag{2}
$$

To simplify the text we will use the following notation. $(.,.)$ and $\|.\|$ are $L^2(\Omega)$ scalar product and norm, $|.|_1$ and $\|.\|_1$ are $H^1(\Omega)$ seminorm and norm. Let us define bilinear form

$$
a(u,v) = \varepsilon(\nabla u, \nabla v) + (b \cdot \nabla u + cu, v). \tag{3}
$$

**Definition 1.** *We say that the function $u \in L^2(0,T,H^1_0(\Omega))$ with the time derivative $\frac{\partial u}{\partial t} \in L^2(0,T,H^{-1}(\Omega))$ is the weak solution of (1), if the following conditions are satisfied*

$$
\begin{aligned}
\left(\frac{\partial u(t)}{\partial t}, v\right) + a(u(t),v) &= (f(t),v) \quad \forall t \in (0,T), \forall v \in H^1_0(\Omega), \\
u(0) &= u^0.
\end{aligned} \tag{4}
$$

It is possible to show that the solution has in general boundary layer around the border of $\Omega$ at $x = 1$ and $y = 1$. Assuming sufficiently compatible data we can avoid the existence of interior layers, which enables us to concentrate on the boundary layers only, see [9] or [4]. Moreover, it is possible to guarantee the S–decomposition