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# A NUMERICAL ALGORITHM FOR SET-POINT REGULATION OF NON-LINEAR PARABOLIC CONTROL SYSTEMS

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**Abstract.** In this paper we hope to draw attention to a particularly simple and extremely flexible design strategy for solving a wide class of "set-point" regulation problems for nonlinear parabolic boundary control systems. By this we mean that the signals to be tracked and disturbances to be rejected are time independent. The theoretical underpinnings of our approach is the well known regulator equations from the geometric theory of regulation applicable in the neighborhood of an equilibrium. The most important point of this work is the wide applicability of the design methodology. In the examples we have employed unbounded sensing and actuation but the method works equally well for bounded input and output operators and even finite dimensional nonlinear control systems. Our examples include: multi-input multi-output regulation for a boundary controlled viscous Burgers' equation; control of a Navier-Stokes flow in two dimensional forked channel; control problem for a non-Isothermal Navier-Stokes flow in two dimensional box domain. Along the way we provide some discussion to demonstrate how the method can be altered to provide many alternative control mechanisms. In particular, in the last section we show how the method can be adapted to solve tracking and disturbance rejection for piecewise constant time dependent signals.

**Key words.** Boundary Control System, Center Manifold, Regulator Equations.

## 1. Introduction

In control theory, regulation of a control system is a fundamental problem that has received considerable attention in the engineering literature. Specific examples of regulation problems include the design of control laws that achieve tracking and disturbance rejection. Our interest in this paper is to present a straightforward methodology for numerical implementation of a strategy (based on the geometric theory of regulation) for solving a wide variety of regulation problems for linear and nonlinear distributed parameter systems with quite general control and sensing, including boundary control and sensing. In the geometric theory of regulation, problems of output regulation include asymptotic tracking of reference signals and rejection of unwanted disturbances. This methodology was first studied by B. Francis [8] and many others in the finite dimensional linear case. In a series of tremendously inspiring papers in the early 1990s, C. I. Byrnes and A. Isidori [11, 12, 13] extended the geometric theory to nonlinear finite dimensional systems. Byrnes and Isidori's work was based on center manifold theory and reduced the design problem to solving a pair of nonlinear operator equations referred to as the Regulator Equations. These equations are also often called the FBI equations after Francis, Byrnes and Isidori. Until recently, a major obstacle to the practical implementation of the method was the inherent difficulty in solving the Regulator Equations. In this area we have made significant progress toward obtaining approximate numerical solutions by developing methods for solving the nonlinear regulator equations. Our techniques have lead to the ability to design control laws even for such complicated systems as the two dimensional Boussinesq approximation of non-isothermal incompressible flows.

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The explicit examples presented in this work (Section 4 and Section 5) are concerned with applications from distributed parameter control in which the plant is given in terms of a nonlinear parabolic partial differential equation. The main reason for our choice of examples is that they provide the most challenging types of examples. In particular, for systems governed by partial differential equations it is not only possible for the state operator to be unbounded but also the input and output maps as well. Here, the expression unbounded means discontinuous. For example, output mappings defined by point evaluation at points or on lower dimensional hypersurfaces either inside the spatial domain or on the boundary of this domain provide discontinuous (unbounded) mappings in the standard $L^2$ energy Hilbert space. Indeed, in the terminology of functional analysis such operators may not even be closable. Similarly, control inputs that enter through the boundary or inside the spatial domain at points or on lower dimensional hypersurfaces are also described by distributional operators which are unbounded mappings in the standard $L^2$ Hilbert space.

It would have been easier to include examples with bounded input and output maps and even examples from finite dimensional control theory since the basic methodology described in this work applies equally well to linear or nonlinear regulation control problems for these types of systems. We hope that the interested reader can easily adapt the roadmap presented here to solve problems for other types of set point control problems.

As we have already mentioned this paper is concerned with set-point regulation problems. These are problems in which the reference signs to be tracked and disturbances to be rejected are independent of time. We focus on this particular class of problems since our numerical algorithm for solving the regulator equations in this case requires a considerably different and much simpler approach than is needed in the more general case of tracking and rejecting time varying signals. The more general case will be the subject of a forthcoming paper [1].

As a disclaimer, in this work we do not investigate the main mathematical properties of the pde models appearing in our application examples. In our opinion such a diversion would seriously detract from the main point of the work which is to exhibit the utility of the design methodology and its numerical implementation. So, for example, we do not go into any details concerning such things as Hilbert space formulations of weak solutions, Sobolev spaces and elliptic estimates needed to guarantee existence and regularity of solutions. To do so would require us to significantly limit the number of examples presented and is not the main point in the work.

The paper is organized as follows. In Section 2 we present the necessary notation and definitions for the general abstract control problem. We briefly discuss the issues related to bounded and unbounded formulations (i.e., boundary type control) and remark that their equivalence have been examined in works such as [15, 16]. In Section 3 we describe the main problem of interest in this work, Problem 3.1. This subsection also contains the details of the design strategy, which derive from the geometric theory of regulation. Our main assumptions, based on the geometric theory of regulation, are captured in Assumptions 3.1 and 3.2. Providing these assumptions are satisfied for a particular control model, it is clear that Problem 3.1 is solvable (at least locally). Section 4 begins with what we consider the most important part of this work, the numerical examples that exhibit the utility of the design strategy presented in Section 3. In Section 5 we provide a general method for tracking and rejecting piecewise constant time dependent signals. The method is