INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 11, Number 1, Pages 86–101

ANALYSIS AND FINITE ELEMENT APPROXIMATION OF BIOCONVECTION FLOWS WITH CONCENTRATION DEPENDENT VISCOSITY

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(Communicated by HONG WANG)

Abstract. The problem of a stationary generalized convective flow modelling bioconvection is considered. The viscosity is assumed to be a function of the concentration of the micro-organisms. As a result the PDE system describing the bioconvection model is quasilinear. The existence and uniqueness of the weak solution of the PDE system is obtained under minimum regularity assumption on the viscosity. Numerical approximations based on the finite element method are constructed and error estimates are obtained. Numerical experiments are conducted to demonstrate the accuracy of the numerical method as well as to simulate bioconvection pattern formations based on realistic model parameters.

Key words. bio-convection, nonlinear partial differential equations, finite element method

1. Introduction

Bio-convection occurs due to on average upwardly swimming micro-organisms which are slightly denser than water in suspensions. A fluid dynamical model treating the micro-organisms as collections of particles was first derived independently by M.Levandowsky, W. S. Hunter and E. A. Spiegel [16], and Y. Moribe [22] which we describe as follows. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. At point $x \in \Omega$, let $\mathbf{u}(x) = \{\mathbf{u}_j(x)\}_{j=1}^3$ and $p(x)$ respectively denote the velocity and pressure of the culture fluid while $c(x)$ refers to the concentration of the micro-organisms. The steady state system for $(\mathbf{u}, c, p)$ takes the form

$$
\begin{aligned}
- \mathbf{div}\ (\nu(c)D(\mathbf{u})) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = -g(1+\gamma c)i_3 + \mathbf{f}\,, &\quad \text{in } \Omega\,,\\
\mathbf{div}\ \mathbf{u} = 0\,, &\quad \text{in } \Omega\,,\\
-\theta\Delta c + \mathbf{u}\cdot\nabla c + U\frac{\partial c}{\partial x_3} = 0\,, &\quad \text{in } \Omega\,.
\end{aligned}
\tag{1.1}
$$

Here $\nu(\cdot) > 0$, as a function of the concentration c, denotes the kinematic viscosity of the culture fluid, $D(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ denotes the stress tensor, $\mathbf{f}$ refers to the volume-distributed external force, g is the acceleration of gravity, θ and U are the diffusion rate and the mean velocity of upward swimming of the micro-organisms respectively, $i_3 = (0, 0, 1)$ is the vertical unitary vector, and the constant $\gamma > 0$ is given by $\gamma = \rho_0/\rho_m - 1$, where ρ_0 is the density of the micro-organisms and ρ_m is the density of the culture fluid.

Received by the editors October 28, 2012 and in revised form January 24, 2013.
2000 *Mathematics Subject Classification.* 35, 65N30, 92.
Y. Cao was partially supported by the National Science Foundation under grand number DMS0914554 and US Air Force Office of Scientific Research under grand number FA550-12-1-0281 and by Guangdong Provincial Government of China through the Computational Science Innovative Research Team program and Guangdong Province Key Laboratory of Computational Science at the Sun Yat-sen University.

The bioconvection model (1.1) is a special case of a more general equation describing the diffusion and transformation of an admixture in a region [1]. The first equation is a Navier-Stokes type equation describing the motion of the viscous micro-organisms while the second equation describes the incompressibility of the culture fluid. The last equation of (1.1) describes the mass conservation:

$$\frac{d}{dt}c + \mathbf{div}\ q = 0\,, \quad \text{in } \Omega\,,$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + (u, \nabla)$ is the material derivative along the fluid particle and $q = -\theta\nabla c + Uci_3$ represents the flux of micro-organisms. We prescribe the boundary conditions for $\mathbf{u}$ and c as

$$\begin{aligned} \mathbf{u} &= 0\,, \quad \text{on } \partial\Omega\,, \\ \theta\frac{\partial c}{\partial \mathbf{n}} - Ucn_3 &= 0\,, \quad \text{on } \partial\Omega\,. \end{aligned} \tag{1.2}$$

The second equation of (1.2) refers to zero flux on the boundary where $\mathbf{n} = (n_1, n_2, n_3)$ is the exterior unitary normal vector on $\partial\Omega$. We further assume the fixed total mass for the micro-organisms:

$$\frac{1}{|\Omega|}\int_\Omega c(x)dx = \alpha\,, \tag{1.3}$$

for some constant α. Condition (1.3) assures that no micro-organisms are allowed to leave or enter the container. Now the complete system describing the motion of micro-organisms takes the form

$$\begin{cases} -\ \mathbf{div}\ (\nu(c)D(\mathbf{u})) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = -g(1+\gamma c)i_3 + \mathbf{f}\,, \quad \text{in } \Omega\,, \\ \mathbf{div}\ \mathbf{u} = 0\,, \quad \text{in } \Omega\,, \\ -\theta\Delta c + \mathbf{u}\cdot\nabla c + U\frac{\partial c}{\partial x_3} = 0\,, \quad \text{in } \Omega\,, \\ \mathbf{u} = 0\,, \quad \theta\frac{\partial c}{\partial \mathbf{n}} - Ucn_3 = 0\,, \quad \text{on } \partial\Omega\,, \\ \frac{1}{|\Omega|}\int_\Omega c(x)dx = \alpha\,. \end{cases} \tag{1.4}$$

In an ideal Newtonian fluid, the viscosity ν is a constant. In this case, the existence of the solution as well as the positivity of the concentration are proved in [14] where the authors considered both the stationary and evolutionary cases. The evolutionary case of system (1.1) with constant viscosity ν is studied numerically in [12]. The numerical study of slightly different bioconvection models can be found in [4], [8], [9], [7] and [13].

In general, for particle models, the viscosity is related to the concentration of the solute. Albert Einstein showed in his Ph.D thesis [6] that

$$\frac{\nu}{\nu_0} = 1 + \xi c \tag{1.5}$$

when the concentration c is small, where ν is the viscosity of the suspension, ν_0 is the viscosity of the pure solution and ξ is a proportionality coefficient, often chosen to be 2.5. This model was later extended by adding a quadratic term of c by Batchelor [2] for larger c ($\geq 10\%$). When the concentration is much higher, the relative viscosity $\frac{\nu}{\nu_0}$ varies as an exponential function of concentration c ([17], [15] and [3]). A recent work [5] showed the existence and uniqueness of a periodic