INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 11, Number 1, Pages 102–116

IMAGE DENOISING BASED ON THE SURFACE FITTING STRATEGY

ZHI-FENG PANG AND BAOLI SHI

Abstract. In this work, we propose a surface fitting strategy based on a two-step model to remove noise from digital images. In the first step, we minimize the total variation energy functional of an image by using the projection gradient method in order to obtain the dual variable as the smoothed normal vector. In the second step, we try to find a surface as the recovered image to fit the smoothed normal vector. Based on the projection gradient method and the variable splitting method, we propose an efficient numerical method to solve this two-step model and also give the convergence analysis of the proposed method. Some numerical comparisons are given to validate the effectiveness of our proposed model.

Key words. Two-step model, Staircase effect, Projection gradient method, Edge indicator function

1.. Introduction

The existence of noise is inevitable in the course of obtaining images. It may be introduced in many different ways, such as image formation processing, image recording and image transmission. These random distortions make it difficult to carry out any required picture processing. Therefore, noise removal is an important and challenging problem in image restoration.

In the view of mathematics, the denoising problem can be expressed as follows: Assume that $g : \Omega \subset R^2 \to R$ denotes a noisy image and u denotes the desired clean image, it follows that

$$g = u + \eta,$$

where η is the additive noise. The aim is to recover the true image u from g.

In practice we want to preserve image edges and features while removing noise for the image denoising problem. Many researchers have devoted their efforts to this study, see [4, 15]. Wherein many variational models have been proposed to eliminate noise and to preserve edges and the small scale characteristics at the same time. The total variation (TV) minimization, as a classical variational model, was first introduced by Rudin, Osher and Fatemi (called the ROF model) in [27] as the following form

$$\min_u \int_\Omega |Du| + \frac{\lambda}{2}\|u - g\|^2_{L^2(\Omega)}, \tag{1.1}$$

where the regularization parameter $\lambda > 0$. It has been demonstrated to be very successful in edge-preserving for image restoration problem, see [10, 14, 18, 19, 20, 22, 27, 28] and references therein. However, the ROF model has the undesirable staircase effect since the smooth regions of the restored image are transformed into the piecewise constants. To overcome this deficiency, some higher-order PDEs [14, 21, 23, 29] have been proposed during the last few years. It has been proved

Received by the editors September 7, 2011 and, in revised form, January 25, 2013.
2000 *Mathematics Subject Classification.* 94A08.

that the higher-order PDEs can effectively alleviate the undesirable staircase effect such as the model [21] with the form

$$\min_u \int_\Omega |D^2 u| + \frac{\zeta}{2}\|u-g\|^2_{L^2(\Omega)}, \tag{1.2}$$

where the tuning parameter $\zeta > 0$. However, the higher-order PDEs are generally complicated to be implemented and can not preserve the edges well.

Recently, a two-step method has been proposed by Lysaker-Osher-Tai (called the LOT model) [22] to reduce staircase effect while preserving important features, such as edges and textures. This two-step model can be denoted as the following two steps:

- In the first step, they minimized the total variation of the unit normal vector $\mathbf{n}$ by the following scheme

$$\min_{|\mathbf{n}|=1} \int_\Omega |\nabla \mathbf{n}| \mathrm{d}x + \frac{\alpha}{2}\|\mathbf{n}-\mathbf{n}_0\|^2_{L^2(\Omega)}, \tag{1.3}$$

  where $\mathbf{n}_0 = \frac{\nabla g}{\|\nabla g\|_2}$ and the regularization parameter $\alpha > 0$.
- In the second step, in order to find a surface to fit the above smoothed normal vector $\mathbf{n}$, they considered

$$\min_v \int_\Omega \left(|\nabla v| - (\nabla v)^T \cdot \mathbf{n}\right) \mathrm{d}x + \frac{\beta}{2}\|v-g\|^2_{L^2(\Omega)}, \tag{1.4}$$

  where the regularization parameter $\beta > 0$.

In the view of the numerical implementation, the proposed algorithms of the LOT model in [22] are complicated and slow because of computing three nonlinear second-order PDEs. Additionally, when the information about noise is not known, this model can not preserve edges or textures well. Therefore, an improved LOT model was suggested in [18]. By letting $\mathbf{n} = (\cos\theta, \sin\theta)^T$ and $\mathbf{n}_0 = (\cos\theta_0, \sin\theta_0)^T$, they used the relationship $|\nabla \mathbf{n}| = |\nabla \theta|$ to transform the minimization problem (1.3) into the following form

$$\min_\theta \int_\Omega |\nabla \theta| \mathrm{d}x + \alpha \int_\Omega (1-\cos(\theta-\theta_0))\mathrm{d}x.$$

Based on the fact that

$$1-\cos(\theta-\theta_0) = 2\sin^2\left(\frac{\theta-\theta_0}{2}\right) \sim \frac{(\theta-\theta_0)^2}{2}, \tag{1.5}$$

i.e. $1-\cos(\theta-\theta_0)$ is the equivalent infinitesimal quantity of $\frac{(\theta-\theta_0)^2}{2}$, they then solved the following problem

$$\min_\theta \int_\Omega |\nabla \theta| \mathrm{d}x + \frac{\alpha}{2}\|\theta-\theta_0\|^2_{L^2(\Omega)} \tag{1.6}$$

in the first step, where θ and θ_0 are the polar angle of $\mathbf{n}$ and $\mathbf{n}_0$ respectively. But only when $(\theta-\theta_0) \to 0$, the formula (1.5) comes into the existence. Actually, the unit normal vector $\mathbf{n}$ of the restoration image u and $\mathbf{n}_0$ defined in the problem (1.3) are impossible to be very close each other. Thus, the scheme substituting $1-\cos(\theta-\theta_0)$ by $\frac{(\theta-\theta_0)^2}{2}$ is not convincing in [18]. In the second step, they introduced an edge indicator function $I(x)$ and adopted the L^1 norm for the image fidelity term. Then this step is given by

$$\min_v \int_\Omega I(x)\left(|\nabla v| - (\nabla v)^T \cdot \mathbf{n}\right) \mathrm{d}x + \gamma|v-g|_{L^1(\Omega)}, \tag{1.7}$$