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# GLOBAL CONVERGENCE OF *A POSTERIORI* ERROR ESTIMATES FOR THE DISCONTINUOUS GALERKIN METHOD FOR ONE-DIMENSIONAL LINEAR HYPERBOLIC PROBLEMS

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**Abstract.** In this paper we study the global convergence of the implicit residual-based *a posteriori* error estimates for a discontinuous Galerkin method applied to one-dimensional linear hyperbolic problems. We apply a new optimal superconvergence result [Y. Yang and C.-W. Shu, *SIAM J. Numer. Anal.*, 50 (2012), pp. 3110-3133] to prove that, for smooth solutions, these error estimates at a fixed time converge to the true spatial errors in the $L^2$-norm under mesh refinement. The order of convergence is proved to be $k+2$, when $k$-degree piecewise polynomials with $k \geq 1$ are used. As a consequence, we prove that the DG method combined with the *a posteriori* error estimation procedure yields both accurate error estimates and $\mathcal{O}(h^{k+2})$ superconvergent solutions. We perform numerical experiments to demonstrate that the rate of convergence is optimal. We further prove that the global effectivity indices in the $L^2$-norm converge to unity under mesh refinement. The order of convergence is proved to be 1. These results improve upon our previously published work in which the order of convergence for the *a posteriori* error estimates and the global effectivity index are proved to be $k+3/2$ and $1/2$, respectively. Our proofs are valid for arbitrary regular meshes using $P^k$ polynomials with $k \geq 1$ and for both the periodic boundary condition and the initial-boundary value problem. Several numerical simulations are performed to validate the theory.

**Key words.** Discontinuous Galerkin method; hyperbolic problems; superconvergence; residual-based *a posteriori* error estimates.

## 1. Introduction

In this paper we analyze a residual-based *a posteriori* error estimates of the spatial errors for the semi-discrete discontinuous Galerkin (DG) method applied to the following one-dimensional linear hyperbolic conservation laws

$$u_t + cu_x = f(x,t), \quad x \in [a,b],\ t \in [0,T],\ c > 0, \tag{1.1a}$$

subject to the initial condition

$$u(x,0) = u_0(x), \quad x \in [a,b], \tag{1.1b}$$

and to either the Dirichlet boundary condition

$$u(a,t) = g(t), \quad t \in [0,T], \tag{1.1c}$$

or to the periodic boundary condition

$$u(a,t) = u(b,t), \quad t \in [0,T]. \tag{1.1d}$$

Here $c > 0$ is a constant speed and $[0,T]$ is a finite time interval. In this paper, we consider, without loss of generality, (1.1) with $c = 1$. In our analysis we select

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the initial and boundary conditions and the source, $f(x,t)$, such that the exact solution, $u(x,t)$, is a smooth function on $[a,b] \times [0,T]$.

The DG method considered here is a class of finite element methods using completely discontinuous piecewise polynomials for the numerical solution and the test functions. DG method combines many attractive features of the classical finite element and finite volume methods. It is a powerful tool for approximating some partial differential equations which model problems in physics, especially in fluid dynamics or electrodynamics. In particular, it provides an appealing approach to address problems having discontinuities, such as those that arise in hyperbolic conservation laws. DG method was initially introduced by Reed and Hill in 1973 as a technique to solve neutron transport problems [30]. In 1974, LaSaint and Raviart [29] presented the first numerical analysis of the method for a linear advection equation. Since then, DG methods have been used to solve ordinary differential equations [5, 18, 28, 29], hyperbolic [14, 15, 16, 17, 23, 24, 26, 27] and diffusion and convection-diffusion [12, 13, 31] partial differential equations. Consult [22] and the references cited therein for a detailed discussion of the history of DG method and a list of important citations on the DG method and its applications.

In recent years, the study of superconvergence and *a posteriori error* estimates of DG methods has been an active research field in numerical analysis. *A posteriori* error estimators employ the known numerical solution to derive estimates of the actual solution errors. They are also used to steer adaptive schemes where either the mesh is locally refined ($h$-refinement) or the polynomial degree is raised ($p$-refinement). For an introduction to the subject of *a posteriori* error estimation see the monograph of Ainsworth and Oden [9]. A knowledge of superconvergence properties can be used to ($i$) construct simple and asymptotically exact *a posteriori* estimates of discretization errors like the one considered in this paper and ($ii$) help detect discontinuities to find elements needing limiting, stabilization and/or refinement. Superconvergence properties for DG methods have been studied in [5, 8, 25, 29] for ordinary differential equations, [4, 10, 5, 7, 21, 32] for hyperbolic problems and [2, 3, 6, 7, 11, 19, 20, 21] for diffusion and convection-diffusion problems.

The first superconvergence result for standard DG solutions of ordinary differential equations appeared in Adjerid *et al.* [5]. They proved that the $k$-degree DG solution of $u' - au = 0$ is $\mathcal{O}(h^{k+2})$ superconvergent at the roots of $(k+1)$-degree right Radau polynomial. Numerical computations indicate that these superconvergence results extend to DG solutions of transient convection problems. However no analysis has been carried out for these results. Later, Cheng and Shu [21] studied the superconvergence property for the DG methods for solving one-dimensional time-dependent linear conservation laws. They proved superconvergence towards a particular projection of the exact solution when the upwind flux is used. The order of superconvergence is proved to be $k + 3/2$, when $k$-degree piecewise polynomials with $k \geq 1$ are used. However, the superconvergence rate obtained in [21] is not optimal. Adjerid and Baccouch [4] investigated the global convergence of the implicit residual-based *a posteriori* error estimates of Adjerid *et al.* [5]. They applied the superconvergence results of Cheng and Shu [21] and proved that these estimates at a fixed time $t$ converge to the true spatial error in the $L^2$-norm under mesh refinement. The order of superconvergence is proved to be $k + 3/2$. They further proved that the global effectivity indices converge to unity at $\mathcal{O}(h^{1/2})$ rate. In this paper, we improve upon the result in [4]. A new optimal superconvergence