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NUMERICAL MODELING OF NON-NEWTONIAN VISCOPLASTIC FLOWS: PART II. VISCOPLASTIC FLUIDS AND GENERAL TRIDIMENSIONAL TOPOGRAPHIES

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Abstract. A new reduced model for the shallow tridimensional viscoplastic fluid is presented in this paper, allowing for the first time an arbitrarily topography. A new numerical approach is also proposed in order to catch efficiently the long-time behavior of the flow and the arrested state. In order to support varying and large time steps, a fully implicit and second order method (BDF2) is proposed. It is combined with an auto-adaptive mesh feature for catching accurately the evolution of front position. This approach was tested on two flows experiments and compared to experimental measurements. The first study shows the efficiency of this approach when the shallow flow conditions are fully satisfied while the second one points out the limitations of the reduced model when these conditions are not fulfilled.

Key words. fluid mechanics ; non-Newtonian fluid ; Bingham model ; asymptotic analysis ; shallow water theory

1. Introduction

The study of shallow flows is motivated by numerous environmental and industrial applications. For Newtonian fluids, this problem was first motivated by hydraulic engineering applications. In 1887, Barré de Saint-Venant [29] introduced for fast Newtonian flows the shallow water approximation, driven by inertia terms while viscous effects are neglected. The original technique, based on an averaged flow-depth, has been extended to the more general asymptotic expansion method. It leads to the same governing equation at zeroth expansion order, but provides a more general theoretical framework for the derivation of reduced models. More recently, slower Newtonian flows [19] and the effect of viscous terms [13] were investigated. But only the more complex non-Newtonian case approaches the complexity of both the manufacturing processes (concretes, foods) and the environmental applications (e.g. mud flows [12, 20], volcanic lava [14, 35], dense snow avalanches [2] or submarine landslides [15]). Concerning non-Newtonian rheologies, shallow approximations of the dam break problem were first studied for a viscoplastic fluid by Lui and Mei [21] and revisited by Balmforth and Craster [6]. See [8, 3] for recent reviews on this subject and [1] for some recent theoretical avances. One may also note the recent interest for the Bostwick consistometer used in food industry [27, 7]. The 2D horizontal dam break problem was used as a benchmark test: the nonlinear reduced equation obtained by the asymptotic method in the shallow limit does not admit an explicit solution and composite [18] or autosimilar solutions [17, 5] were proposed instead (see also [4]).

Thus, a direct numerical resolution without any simplification is of the utmost interest to fully solve such a nonlinear problem. Let us mention the computation of the arrested state [23] by a specific finite difference scheme. Nevertheless, the proposed numerical procedure is based on some specific features of the solution of the horizontal 1D dam break problem and does not extend to a more general

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situation, such as non-constant slopes or 3D topographies. Some authors explored specific 3D topographies by using specific axisymmetric coordinate systems such as a curved channel [24] or a conical surface [36]. These authors used a finite difference discretization scheme and then, an alternating direction algorithm for solving the resulting algebraic nonlinear set of equations. This numerical approach was next reused in [8] for similar 3D computations on a flat inclined topography. It is important to note that all previous reduced models was developed for some specific topography and not reusable for another one.

The aim of this paper is to bring a new robust and efficient numerical method for the resolution of the shallow approximation of 3D viscoplastic flow problem on a general topography. Numerical results obtained with the present model are validated by comparisons with previous computations on specific topographies. The proposed numerical algorithm for solving the problem extends a previous numerical work preformed on the horizontal 2D dam break problem [34]. The present numerical scheme provides a fully automatic space-adaptive feature which enables an accurate capture of the evolution of front position and which is also able to predict accurately the long-time behavior and the arrested state of the model.

This manuscript has been divided as follow: Section 2 introduces the problem statement and the reduced problem obtained after the asymptotic analysis under the shallow flow approximation. Section 3 develops details of the numerical resolution of this nonlinear problem. Section 4 presents the numerical results and two comparisons between the present theory and experiment measurements available in the literature.

2. The reduced problem for a general 3d topography

2.1. Problem statement. The Herschel-Bulkley [16] constitutive equation expresses the deviatoric part τ of the stress tensor versus the rate of deformation tensor $\dot{\gamma}$ as:

$$\left\{ \begin{array}{ll} \tau = K|\dot{\gamma}|^{n-1}\dot{\gamma} + \tau_y \frac{\dot{\gamma}}{|\dot{\gamma}|} & \text{when } \dot{\gamma} \neq 0, \\ |\tau| \leq \tau_y & \text{otherwise.} \end{array} \right. \tag{1}$$

where $K > 0$ is the consistency, $n > 0$ is the power-law index and τ_y is the yield stress. Here $|\tau| = ((1/2)\sum_{i,j=1}^{3} \tau_{ij}^2)^{1/2}$ denotes the conventional norm of a symmetric tensor in mechanics. The total Cauchy stress tensor is $\sigma = -p.I + \tau$ where p is the pressure and I the identity tensor. When $n = 1$ and $\tau_y = 0$, the fluid is Newtonian and K is the viscosity. For a general $n > 1$ and when $\tau_y = 0$, the model describes a power-law fluid. When $n = 1$ and $\tau_y \geq 0$, this model reduces to the Bingham one [9]. The constitutive equation (1) is completed by the conservations of momentum and mass:

$$\rho\left(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\right) - \mathbf{div}(-p.I + \tau) = \rho\mathbf{g}, \tag{2}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{3}$$

where $\rho > 0$ is the constant density and $\mathbf{g}$ is the gravity vector. There are three equations (1)-(3) and three unknowns τ, $\mathbf{u}$ and p. The corresponding problem is closed by defining the boundary and initial conditions.

The flow over a variable topography is considered (see Fig. 1). For any time $t > 0$, the flow domain is denoted as $Q(t)$. We suppose that $Q(t)$ can be described as:

$$Q(t) = \{(x, y, z) \in \Omega \times \mathbb{R};\ f(x, y) < z < f(x, y) + h(t, x, y)\}$$