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CONVERGENCE OF ADAPTIVE FEM FOR SOME ELLIPTIC OBSTACLE PROBLEM WITH INHOMOGENEOUS DIRICHLET DATA

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Abstract. In this work, we show the convergence of adaptive lowest-order FEM (AFEM) for an elliptic obstacle problem with non-homogeneous Dirichlet data, where the obstacle χ is restricted only by $\chi \in H^2(\Omega)$. The adaptive loop is steered by some residual based error estimator introduced in Braess, Carstensen & Hoppe (2007) that is extended to control oscillations of the Dirichlet data, as well. In the spirit of Cascon et al. (2008), we show that a weighted sum of energy error, estimator, and Dirichlet oscillations satisfies a contraction property up to certain vanishing energy contributions. This result extends the analysis of Braess, Carstensen & Hoppe (2007) and Page & Praetorius (2013) to the case of non-homogeneous Dirichlet data as well as certain non-affine obstacles and introduces some energy estimates to overcome the lack of nestedness of the discrete spaces.

Key words. Adaptive finite element methods, Elliptic obstacle problems, Convergence analysis.

1. Introduction

1.1. Comments on prior work. Adaptive finite element methods based on various types of a posteriori error estimators are a famous tool in science and engineering and are used to deal with a wide range of problems. As far as elliptic boundary value problems are concerned, convergence and even quasi-optimality of the adaptive scheme is well understood and analyzed, see e.g. [5, 16, 19, 29, 30, 37, 38].

In recent years the analysis has been extended and adapted to cover more general applications, such as the p-Laplacian [40], mixed methods [13], non-conforming elements [14], and obstacle problems. The latter is a classic introductory example to study variational inequalities which represent a whole class of problems that often arise in physical and economical context. One major application is the oscillation of a membrane that must stay above a certain obstacle. Other examples are filtration in porous media or the Stefan problem (i.e. melting solids), in both of which non-homogeneous Dirichlet data play an important role. Also in the financial world, obstacle problems arise, e.g. in the valuation of the American put option [34], where one has to deal with various non-affine obstacles. For a broader understanding of these problems, we refer to [21] and the references therein. The great applicability in many scientific areas thus make numerical analysis and mathematical understanding of the obstacle problem both, interesting and important. As far as a posteriori error analysis is concerned, we refer to [6, 8, 9, 17, 26, 31, 39]. Convergence of an adaptive method for elliptic obstacle problems with globally affine obstacle was proven in [10, 33]. Both of these works, however, considered homogeneous Dirichlet boundary data and affine obstacles only.

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In [11], the authors generalized the analysis of [10] to general $H^1(\Omega)$-obstacles. Convergence of the proposed method is, however, only proved up to some consistency errors, and the analysis relies on homogeneous Dirichlet conditions. Moreover, some steps in the analysis are somewhat unclear, as the reliability proof depends on certain estimates which are not explained or properly cited.

1.2. Contributions of current work. We treat the case of a general obstacle $\chi \in H^2(\Omega)$. By a simple transformation and allowing non-homogeneous Dirichlet data (Prop. 4), this can, however, be reduced to the case of a constant zero-obstacle. Since our analysis works for general globally affine obstacles, even without the reduction step, we consider affine obstacles and non-homogeneous Dirichlet data in the following. We follow the ideas from [33], i.e. adaptive P1-FEM for some elliptic obstacle problem with globally affine obstacle. Contrary to [10, 11, 33], however, we allow non-homogeneous Dirichlet boundary data $g \in H^1(\Gamma)$, which are approximated by some g_ℓ via nodal interpolation within each step of the adaptive loop. In contrast to the aforementioned works, we thus do not have nestedness of the discrete ansatz sets, which is a crucial ingredient of the prior convergence proofs. In the spirit of [16] and in analogy to [33], we show that our adaptive algorithm, steered by some estimator ϱ_ℓ, guarantees that the combined error quantity

$$\Delta_\ell := \mathcal{J}(U_\ell) - \mathcal{J}(u_\ell) + \gamma\,\varrho_\ell^2 + \lambda\,\mathrm{apx}_\ell^2 \tag{1}$$

is a contraction up to some vanishing perturbations $\alpha_\ell \to 0$, i.e.

$$\Delta_{\ell+1} \leq \kappa\Delta_\ell + \alpha_\ell, \tag{2}$$

with $0 < \gamma, \kappa < 1, \lambda > 0$, and $\alpha_\ell \geq 0$. The data oscillations on the Dirichlet boundary are controlled by the term apx_ℓ, and the quantity u_ℓ denotes the continuous solution subject to discrete boundary data g_ℓ, which is introduced to circumvent the lack of nestedness of the discrete spaces. Convergence then follows from a weak reliability estimate of ϱ_ℓ, namely

$$\varrho_\ell \to 0 \quad \Rightarrow \quad \|u - U_\ell\|_{H^1(\Omega)} \to 0, \tag{3}$$

since $\varrho_\ell \lesssim \Delta_\ell \to 0$ as $\ell \to \infty$. We point out that our convergence proof makes use of the so called *estimator reduction* and is thus fairly independent of the mesh-refinement strategy. This is an improvement over the earlier works [10, 11] which rely on the *discrete local efficiency* of the underlying estimator and therefore require mesh-refinement strategies which guarantee the so-called *interior node property.*

1.3. Outline of current work. In Section 2, we formulate the continuous model problem and recall its unique solvability. In Section 3, the same is done for the discretized problem. Section 4 is a collection of the main results of this paper. Here, we introduce the error estimator ϱ_ℓ, which is a generalization of the corresponding estimators from [10, 33]. We then state its weak reliability (Theorem 7) and our version of the adaptive algorithm (Algorithm 9). Finally (Theorem 10), we state that the sequence of discrete solutions indeed converges towards the continuous solution $u \in H^1(\Omega)$. The subsequent Sections 5–7 are then devoted to the proofs of the aforementioned results and numerical illustrations.

2. Model Problem

2.1. Problem formulation. We consider an elliptic obstacle problem in $\mathbb{R}^2$ on a bounded Lipschitz domain Ω with polygonal boundary $\Gamma := \partial\Omega$. An obstacle