INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 11, Number 2, Pages 255–270

EXPANDED MIXED FINITE ELEMENT DOMAIN DECOMPOSITION METHODS ON TRIANGULAR GRIDS

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(Communicated by F.J. Gaspar)

This paper is dedicated to Professor Francisco J. Lisbona on the occasion of his 65th birthday

Abstract. In this work, we present a cell-centered time-splitting technique for solving evolutionary diffusion equations on triangular grids. To this end, we consider three variables (namely the pressure, the flux and a weighted gradient) and construct a so-called expanded mixed finite element method. This method introduces a suitable quadrature rule which permits to eliminate both fluxes and gradients, thus yielding a cell-centered semidiscrete scheme for the pressure with a local 10-point stencil. As for the time integration, we use a domain decomposition operator splitting based on a partition of unity function. Combining this splitting with a multiterm fractional step formula, we obtain a collection of uncoupled subdomain problems that can be efficiently solved in parallel. A priori error estimates for both the semidiscrete and fully discrete schemes are derived on smooth triangular meshes with six triangles per internal vertex.

Key words. Cell-centered finite difference, domain decomposition, error estimates, fractional step, mixed finite element, operator splitting.

1. Introduction

We consider a parabolic initial-boundary value problem that models single phase flow in porous media. The problem can be written as a system of two first-order equations of the form

$$p_t + \nabla\cdot\mathbf{u} = f \qquad \text{in } \Omega\times(0,T], \tag{1a}$$

$$\mathbf{u} = -K\nabla p \qquad \text{in } \Omega\times(0,T], \tag{1b}$$

$$p = p_0 \qquad \text{in } \Omega\times\{0\}, \tag{1c}$$

$$p = g \qquad \text{on } \Gamma_D\times(0,T], \tag{1d}$$

$$\mathbf{u}\cdot\mathbf{n} = 0 \qquad \text{on } \Gamma_N\times(0,T], \tag{1e}$$

where $\Omega\subset\mathbb{R}^2$ is a convex polygonal domain with Lipschitz continuous boundary $\partial\Omega = \overline{\Gamma}_D\cup\overline{\Gamma}_N$ such that $\Gamma_D\cap\Gamma_N=\emptyset$. In this formulation, $K\equiv K(\mathbf{x})\in\mathbb{R}^{2\times 2}$ is a symmetric and positive definite tensor satisfying, for some $0<\kappa_*\le\kappa^*<\infty$,

$$\kappa_*\xi^T\xi \le \xi^T K\,\xi \le \kappa^*\xi^T\xi \qquad \forall\,\xi\neq\mathbf{0}\in\mathbb{R}^2, \tag{2}$$

and $\mathbf{n}$ is the outward unit vector normal to $\partial\Omega$. Typically, p represents the fluid pressure, $\mathbf{u}$ is the Darcy velocity and K denotes the hydraulic conductivity tensor.

In this work, we propose and analyze a family of mixed finite element (MFE) time-splitting methods for the solution of problem (1). Via the method of lines approach, the original problem is first reduced to a system of ordinary differential equations using a spatial semidiscretization technique. More precisely, we consider a variant of the standard mixed formulation called the expanded MFE method (cf.

Received by the editors November 12, 2012 and, in revised form, July 7, 2013.
2000 *Mathematics Subject Classification.* 65M06, 65M12, 65M15, 65M20, 65M55, 65M60, 76S05.

[1, 2, 3, 10, 12]). Besides the pressure p and the flux $\mathbf{u}$, this method introduces an additional explicit unknown, namely the adjusted gradient λ. The newly defined variable avoids inverting tensor K, thus allowing for the presence of non-negative conductivities in the flow domain Ω (as a difference, K is assumed to be strictly positive in the standard mixed method). Following [1, 2], we consider the lowest order Raviart–Thomas (RT_0) finite element spaces on triangles (cf. [21]), and subsequently define a suitable quadrature rule that permits to eliminate both $\mathbf{u}$ and λ. As a result, the expanded MFE formulation is reduced to a cell-centered finite difference scheme for the pressure with a local 10-point stencil. In the context of elliptic problems, this idea has been already studied in [6, 8, 16] for the standard mixed method on triangular grids. Similar strategies have been also investigated in the case of rectangular elements (cf. [3, 22, 28]).

The stiff initial value problem resulting from the previous stage is integrated in time by using a domain decomposition splitting technique. This kind of splitting was first introduced in [25, 26] for the construction of regionally-additive schemes and has been subsequently used in [14, 15, 19] for solving linear parabolic problems. In combination with this splitting, we define a family of time integrators belonging to the class of m-part fractional step Runge–Kutta (FSRK$_m$) methods (cf. [9]). Such methods are composed by merging together m diagonally implicit Runge–Kutta schemes into a single composite formula. In particular, we consider the so-called Yanenko's method (cf. [30]), which has been proved to be unconditionally contractive for different splitting functions (see [17, 27]). The fully discrete scheme is thus a collection of uncoupled subdomain problems that can be solved in parallel without the need for Schwarz-type iteration procedures.

The design and analysis of expanded MFE fractional step methods for parabolic problems have been addressed in the earlier works [4, 5]. In both cases, though, the problems were discretized on rectangular meshes using an alternating direction implicit (ADI) technique. In the present paper, we extend the results from these works to the case of domain decomposition splitting methods on triangular grids, thus yielding added flexibility to the resulting algorithms.

The rest of the paper is outlined as follows. In Section 2, we introduce the expanded MFE method and subsequently derive a cell-centered finite difference scheme for the pressure. The convergence analysis of the semidiscrete scheme is described in the next section. Section 4 further presents the family of fractional step time integrators based on a domain decomposition splitting technique. Finally, a priori error estimates for the fully discrete scheme are obtained in Section 5.

2. The expanded mixed finite element method

In order to define an expanded formulation, we need to introduce the additional unknown $\lambda \equiv \lambda(\mathbf{x}, t) = -G^{-1}\nabla p$. This variable is referred to as the adjusted gradient and involves a symmetric and positive definite tensor $G \equiv G(\mathbf{x}) \in \mathbb{R}^{2\times 2}$, to be defined below. In this context, the equation (1b) can be rewritten as

$$G\lambda = -\nabla p \qquad \text{in } \Omega \times (0, T], \tag{3a}$$

$$\mathbf{u} = KG\lambda \qquad \text{in } \Omega \times (0, T]. \tag{3b}$$

These two equations, together with (1a) and the corresponding initial and boundary data, represent the so-called expanded mixed formulation in the triple $(\mathbf{u}, \lambda, p)$.

2.1. The weak formulation. For a domain $R \subset \mathbb{R}^2$, let $W^{k,p}(R)$ be the standard Sobolev space, with $k \in \mathbb{R}$ and $1 \leq p \leq \infty$, endowed with the norm and seminorm $\|\cdot\|_{k,p,R}$ and $|\cdot|_{k,p,R}$, respectively. Let $H^k(R)$ be the Hilbert space $W^{k,2}(R)$,