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NON-HOOKEAN BEAMS AND PLATES: VERY WEAK SOLUTIONS AND THEIR NUMERICAL ANALYSIS

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Dedicated to the 65th birthday of Professor F. J. Lisbona

Abstract. We consider very weak solutions of a nonlinear version (non-Hookean materials) of the beam stationary Bernoulli-Euler equation, as well as the similar extension to plates, involving the bi-Laplacian operator, with Navier (hinged) boundary conditions. We are specially interested in the case in which the usual Sobolev space framework cannot be applied due to the singularity of the load density near the boundary. We present some properties of such solutions as well as some numerical experiences illustrating how the behaviour of the very weak solutions near the boundary is quite different to the one of more regular solutions corresponding to non-singular load functions.

Key words. Beam and plate, non Hookean material, very weak solutions, numerical experiences.

1. Introduction

Given a linear boundary value problem on a bounded regular open set Ω of $\mathbb{R}^N$

$$(P_L)\begin{cases} Lu = f(x) & \text{in } \Omega, \\ + \text{ boundary conditions} \equiv \ (BC) & \text{on } \partial\Omega, \end{cases}$$

where Lu denotes an elliptic differential operator (of order $2m$, $m \in \mathbb{N}$) in divergence form, the usual notion of *weak solution* is defined by introducing the associated "energy space", $V \subset H^m(\Omega)$ (the Sobolev space or order m, i.e. $D^\alpha u \in L^2(\Omega)$ for any $\alpha \in \mathbb{N}^N$, $|\alpha| \leq m$), and then, assumed that

$$f \in V', \tag{1}$$

we introduce the associated bilinear form $a : V \times V \to \mathbb{R}$, and require the condition

$$a(u, \zeta) = \langle f, \zeta \rangle_{V'V}, \text{ for any } \zeta \in V$$

(see [17], [1] and their many references).

A weaker notion of solution can be given leading to a correct mathematical treatment for a more general class of data f (i.e. for f not necessarily in V'). For instance, for $f \in L^1_{loc}(\Omega)$ the notion of *very weak solution of problem* (P_L) can be introduced by integrating $2m-$times by parts (and not merely $m-$times as before) and by requiring, merely, that $u \in L^1(\Omega)$ and that

$$\int_\Omega u(x) L^* \zeta(x) dx = \int_\Omega f(x) \zeta(x) dx,$$

for any $\zeta \in W := \overline{\{\zeta \in C^{2m}(\overline{\Omega}) \colon \zeta \text{ satisfies } (BC)\}}^{W^{2m,\infty}(\Omega)}$, once we assume that

$$\int_\Omega |f(x)\zeta(x)|\, dx < \infty, \text{ for any } \zeta \in W.$$

Here L^* denotes the adjoint operator of L.

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Most of the theory on very weak solutions available in the literature deals with second order equations. Recently, sharper results have been obtained, to this case, when $f \in L^1(\Omega : \delta)$, with $\delta = dist\,(x, \partial\Omega)$. That was originally proved by Haim Brezis, in the seventies, in a famous unpublished manuscript concerning Dirichlet boundary conditions (see also a 1996 paper [5]). For more recent references see [29], [20], [21] and [22]).

The main goal of my past lecture at Jaca 2010 (see [18]) was to present some new results proving that in the case of *higher order* equations the class of $L^1_{loc}(\Omega)$ data for which the existence and uniqueness of a very weak solution can be obtained is, in general, larger than $L^1(\Omega : \delta)$ (the optimal class for the case of second order equations). For instance, for the case of the beam equation with Dirichlet boundary conditions ($u = u' = 0$ on the boundary) I proved that *the optimal class of data* is the space $L^1(\Omega : \delta^2)$ but, for instance, for the simply supported beam ($u = u'' = 0$ on the boundary) *the optimal class of data* is again $L^1(\Omega : \delta)$. One of my main arguments was the use of the Green function $G(x, y)$ associated to the corresponding boundary value problem.

An important open problem in our days is the searching of solutions (beyond the class of weak solutions) for the case in which the operator L is *nonlinear*. Obviously, we cannot integrate $2m-$times by parts and, which seems to be more important, we do not have any kind of Green function associated to the problem.

The main goal of this paper is to present some new results concerning very weak solutions for *nonlinear problems*. Moreover, we shall give here some indications about their numerical approximation. We point out that, without loss of generality we can assume that the beam is represented by the interval $(0, L)$ with $L = 1$ (which we shall do in the rest of the lecture). To fix ideas I will concentrate my attention in the *nonlinear beam equation with simply supported boundaries*

$$(B_{SS}) \begin{cases} \phi(u''(x))'' = f(x) & \text{in } \Omega = (0,1), \\ u(0) = \phi(u'')(0) = 0, \\ u(1) = \phi(u'')(1) = 0, \end{cases}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous strictly increasing function such that $\phi(0) = 0$. A standard example corresponds to the linear case $\phi(s) = EIs$ for any $s \in \mathbb{R}$ (E, I positive constants) but many other cases arise in the more diverse applications (case of non-Hookean materials such as cat iron, stone, rubber, bioelastic materials, concrete and most of the composite materials). Again, by dimensional analysis we can assume equal to one any constant arising in the constitutive law of the material. So, for instance, a very often treated case in the literature is $\phi(s) = |s|^{\alpha-1} s$ for some $\alpha > 0$ (notice that $\alpha = 1$ corresponds to the linear case: see [1]).

We shall also make some few comments on the case of a *nonlinear cantilever beam*

$$(B_{Cant}) \begin{cases} \phi(u''(x))'' = f(x) & \text{in } \Omega = (0,1), \\ u(0) = u'(0) = 0, \\ \phi(u'')(1) = \phi(u'')'(1) = 0. \end{cases}$$

In the last section we shall consider a hinged plate (i.e. with the Navier boundary conditions) or more in general, the N-dimensional problem

$$(P_{Nd}) \begin{cases} -\Delta\phi(-\Delta u(x)) = f(x) & \text{in } \Omega \subset \mathbb{R}^N, \\ u = \phi(-\Delta u) = 0, & \text{on } \partial\Omega. \end{cases}$$