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A FULLY DISCRETE CALDERÓN CALCULUS FOR TWO DIMENSIONAL TIME HARMONIC WAVES

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Dedicated to Francisco 'Paco' Lisbona on the occasion of his 65th birthday

Abstract. In this paper, we present a fully discretized Calderón Calculus for the two dimensional Helmholtz equation. This full discretization can be understood as highly non-conforming Petrov-Galerkin methods, based on two staggered grids of mesh size h, Dirac delta distributions substituting acoustic charge densities and piecewise constant functions for approximating acoustic dipole densities. The resulting numerical schemes from this calculus are all of order h^2 provided that the continuous equations are well posed. We finish by presenting some numerical experiments illustrating the performance of this discrete calculus.

Key words. Calderón calculus, Boundary Element Methods, Dirac deltas distributions, Nyström methods.

1. Introduction

In this paper we present a very simple and compatible Nyström discretization of all boundary integral operators for the Helmholtz equation in a smooth parametrizable curve in the plane. The discretization uses a naïve quadrature method for logarithmic integral equations, based on two staggered grids, and due to Jukka Saranen and Liisa Schroderus [13] (see also [15] and [2]). This is combined with an equally simple staggered grid discretization of the hypersingular operator, recently discovered in [8]. If the displaced grids used for the discretization of these two operators are mutually reversed, then it is possible to combine these two discretizations with a simple minded Nyström method for the double layer operator and its adjoint. The complete set of operators is complemented with a fully discrete version of the single and double layer potentials. We will explain the construction of the discrete set and reinterpret it as a non-conforming Petrov- Galerkin discretization of the operators (using Dirac deltas and piecewise constant functions) to which we apply midpoint integration in every element integral.

Once the semivariational form has been reached we will show inf-sup conditions for all discrete operators involved and consistency error estimates based on asymptotic expansions of the error in the style of [2, 5, 6]. We will finally state and sketch the proof of some convergence error estimates. While some of the results, for individual equations (mainly based on indirect boundary integral formulations) had already appeared in previous papers, this is the first time that the entire Calderón Calculus is presented in its entirety. Let it be emphasized, that this is probably the simplest form of *discretizing simultaneously all the potentials and integral operators* for the Helmholtz equation in the plane and that the methods we obtain are of

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order two. Barring the conceptual difficulty of understanding the boundary integral operators, the methods have the simplicity of basic Finite Difference Methods and require no effort in their implementation: all discrete elements are described in full, natural data structures can be easily figured out from the way the geometry is sampled, and no additional discretization step (quadrature, assembly by element, mapping to a reference element) is required. The methods will be presented for the case of a single curve, but we will hint at its immediate extension to the case of multiple scatterers.

In a final section devoted to numerical experiments, we will show how to use the methods for transmission problems and how to construct combined field integral representations.

2. Calderón calculus for exterior Helmholtz boundary problems

2.1. Potentials and operators. Let Γ be a smooth simple closed curve given by a regular 1-periodic positively oriented parametrization $\mathbf{x} = (x_1, x_2) : \mathbb{R} \to \Gamma \subset \mathbb{R}^2$. Let $\mathbf{n}(t) := (x_2'(t), -x_1'(t))$ be a non-normalized outward pointing normal vector at $\mathbf{x}(t) \in \Gamma$. The domain exterior to Γ will be denoted Ω^+. As a reminder of the fact that we are taking limits from this exterior domain, the superscript $+$ will be used in trace and normal derivative operators.

Let us introduce the exterior Helmholtz equation

$$\Delta U + k^2 U = 0 \text{ in } \Omega^+, \quad \nabla U(\mathbf{z}) \cdot (\tfrac{1}{|\mathbf{z}|}\mathbf{z}) - \imath k U(\mathbf{z}) = o(\tfrac{1}{\sqrt{|\mathbf{z}|}}), \quad \text{as } |\mathbf{z}| \to \infty, \tag{1}$$

where $k > 0$ is the wavenumber. Given 1-periodic complex-valued functions η and ψ, the (parametrized) single and double layer potentials for the Helmholtz equation (1) are defined, respectively, with the formulas

$$\begin{aligned} \big(\mathrm{S}\,\eta\big)(\mathbf{z}) &:= \frac{\imath}{4}\int_0^1 H_0^{(1)}(k|\mathbf{z}-\mathbf{x}(t)|)\eta(t)\,\mathrm{d}t, \\ \big(\mathrm{D}\,\psi\big)(\mathbf{z}) &:= \frac{\imath k}{4}\int_0^1 H_1^{(1)}(k|\mathbf{z}-\mathbf{x}(t)|)\frac{(\mathbf{z}-\mathbf{x}(t))\cdot\mathbf{n}(t)}{|\mathbf{z}-\mathbf{x}(t)|}\psi(t)\,\mathrm{d}t \end{aligned}$$

for arbitrary $\mathbf{z} \in \mathbb{R}^2 \setminus \Gamma$. (Here $H_n^{(1)}$ is the Hankel function of the first kind and order n.) The single and double layer potentials define radiating solutions of the Helmholtz equation for any η, ψ. Moreover, if U is a $\mathcal{C}^1(\overline{\Omega^+})$ solution of (1) and we define

$$\varphi = \gamma^+ U := U|_\Gamma \circ \mathbf{x}, \qquad \lambda = \partial_{\mathbf{n}}^+ U := ((\nabla U)|_\Gamma \circ \mathbf{x}) \cdot \mathbf{n}, \tag{2}$$

then [9, 14]

$$U(\mathbf{z}) = (\mathrm{D}\,\varphi)(\mathbf{z}) - (\mathrm{S}\,\lambda)(\mathbf{z}), \quad \mathbf{z} \in \Omega^+. \tag{3}$$

We note that the representation formula (3), depending on parametrized Cauchy data (2), can be extended to any locally H^1 solution of (1). In this work we will restrict our attention to smooth solutions though.

Associated to the layer potentials we have three integral operators.

$$(\mathrm{V}\eta)(s) := \frac{\imath}{4}\int_0^1 H_0^{(1)}(k|\mathbf{x}(s)-\mathbf{x}(t)|)\eta(t)\,\mathrm{d}t, \tag{4a}$$

$$(\mathrm{K}\psi)(s) := \frac{\imath k}{4}\int_0^1 H_1^{(1)}(k|\mathbf{x}(s)-\mathbf{x}(t)|)\frac{(\mathbf{x}(s)-\mathbf{x}(t))\cdot\mathbf{n}(t)}{|\mathbf{x}(s)-\mathbf{x}(t)|}\psi(t)\,\mathrm{d}t, \tag{4b}$$

$$(\mathrm{J}\eta)(s) := \frac{\imath k}{4}\int_0^1 H_1^{(1)}(k|\mathbf{x}(s)-\mathbf{x}(t)|)\frac{(\mathbf{x}(t)-\mathbf{x}(s))\cdot\mathbf{n}(s)}{|\mathbf{x}(s)-\mathbf{x}(t)|}\eta(t)\,\mathrm{d}t, \tag{4c}$$