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INTERIOR LAYERS IN A SINGULARLY PERTURBED TIME DEPENDENT CONVECTION–DIFFUSION PROBLEM

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This paper is dedicated to Francisco Lisbona, on the occasion of his 65th birthday

Abstract. A linear singularly perturbed time dependent convection–diffusion problem is examined. The initial condition is designed to have steep gradients in the vicinity of the inflow point, which are transported in time, thus creating a moving interior shock layer. The location of this interior layer is tracked by the characteristics of the reduced first order problem. A numerical method is designed and analysed, which consists of a monotone finite difference operator and a piecewise-uniform Shishkin mesh, which is aligned to the characteristic curve emanating from the initial shock location. Parameter explicit error bounds are established and numerical results are presented to illustrate the performance of the numerical method.

Key words. Singular perturbation, interior layer, Shishkin mesh.

1. Introduction

Standard numerical algorithms for partial differential equations are inefficient when steep gradients are present in the solution. Steep gradients arise naturally in the solutions of singularly perturbed problems and in problems where the data (coefficients, source term, boundary/initial conditions, boundary of the domain) are not smooth. Problems with incompatibilities between the initial and boundary data in parabolic problems arise, for example, in mathematical models in poroelasticity [4]. In particular, interior layers can appear, in the solution of a singularly perturbed convection-diffusion parabolic problem, throughout the entire domain if the initial function and the boundary condition do not coincide at the inflow corner point. An alternative mathematical model could be considered where such an incompatibility is regularized by an initial function which is itself the solution of a singularly perturbed ordinary differential equation. In this paper, we construct and analyse a numerical method for a class of singularly perturbed convection–diffusion parabolic problems, where the solution has steep gradients both internally and in a small neighbourhood of the inflow corner point.

This paper is a companion paper to [5], where the following class of problems was studied: Find $\hat{u}$ that satisfies the singularly perturbed differential equation

$$\hat{L}_\varepsilon \hat{u} := -\varepsilon \hat{u}_{ss} + \hat{a}(t)\hat{u}_s + \hat{u}_t = \hat{f}(s,t), \quad (s,t) \in Q := (0,1) \times (0,T], \tag{1a}$$

$$\hat{a}(t) \geq \alpha > 0, t \geq 0; \tag{1b}$$

and the boundary and initial conditions

$$\hat{u}(0,t) = \hat{\phi}_L(t),\ \hat{u}(1,t) = \hat{\phi}_R(t), \quad 0 < t \leq T, \tag{1c}$$

$$\hat{u}(s,0) = \hat{\phi}(s;\varepsilon), \quad 0 \leq s \leq 1. \tag{1d}$$

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The initial condition $\hat{\phi}$ is smooth, but contains an interior layer (see (3)) in the vicinity of a point $s = d, 0 < d < 1$, with d independent of ε. The characteristic curve associated with the reduced differential equation (formally set $\varepsilon = 0$ in (1)) can be described by the set of points

$$\Gamma^* := \{(\gamma(t), t) |\, \gamma'(t) = \hat{a}(t),\ 0 < \gamma(0) = d < 1\}, \tag{2a}$$

that partitions the domain $\bar{Q}$ into two subdomains either side of Γ^*

$$\bar{Q}^- := \{(s, t) |\, s \le \gamma(t) \le 1,\, 0 \le t \le T\}, \tag{2b}$$

$$\bar{Q}^+ := \{(s, t) |\, s \ge \gamma(t) \ge 0,\, 0 \le t \le T\}. \tag{2c}$$

The solution of problem (1) will have an interior layer of width $O(\sqrt{\varepsilon})$ (emanating from the initial condition) which moves in time along Γ^*. In general a boundary layer of width $O(\varepsilon)$ will also appear in the vicinity of the edge $x = 1$. We restrict the size of the final time T so that the interior layer does not interact with this boundary layer. Since $\hat{a} > 0$, the function $\gamma(t)$ is monotonically increasing. Thus, we limit the final time T such that $0 < c < 1 - \gamma(T)$. The parabolic problem examined in [5] may be viewed as a regularization of a singularly perturbed parabolic problem with a discontinuous initial condition.

In this paper, we consider the effect of an interior layer forming initially within a distance greater than or equal to $C\varepsilon^p$, with $p < 1/2$, of the corner $(0, 0)$. The main differences between this paper and [5] occur to the left (in the subdomain $\bar{Q}^-$) of the interior layer. On the right side of the interior layer (in the subdomain $\bar{Q}^+$), the numerical method is essentially identical to what was reported in [5]. Both the numerical method and its associated numerical analysis are different in $\bar{Q}^-$ to what was presented in [5]. The parabolic problem examined in this paper may be viewed as a regularization of a singularly perturbed parabolic problem where the inflow boundary condition and the initial condition do not agree at the inflow corner. Although the solution of the regularized problem does not approximate the solution of a problem with an incompatibility in the vicinity of the inflow corner point $(0, 0)$, the regularized problem may be of interest to researchers interested in simulating the creation of a travelling interior layer, such as can be seen in Figure 3 in §5.2.

In this paper, the time derivatives of the interior layer component (denoted here by $\hat{z}^-$) depend adversely on d, which in turn depends on ε. To construct a parameter-uniform numerical method, the region $\bar{Q}^-$ is further decomposed into two subregions. The first subregion of $\bar{Q}^-$ is designed so that a fine mesh is aligned to the characteristic curve Γ^*. The second subregion is what remains in $\bar{Q}^-$ after this first subregion has been identified. Within each subregion, a particular coordinate system is utilized so that the discretization within each subregion takes place on a rectangular mesh.

In §2, the solution of the continuous problem is decomposed into a sum of components and parameter explicit bounds on each of these components are established. A coordinate transformation, related to the characteristic curve Γ^*, is introduced and the time derivatives of the interior layer component are shown to be bounded in this transformed coordinate system. In §3, the discrete problem is constructed, which involves a piecewise-uniform Shishkin mesh. The numerical method is analysed in §4 and some numerical results are presented in §5.1 to illustrate the theoretical error bounds.

The theoretical analysis in this paper requires that the location of the interior layer in the initial condition is bounded away from the corner $(0, 0)$ by a distance $d > C\sqrt{\varepsilon}$. In §5.2, we present numerical results for the numerical method presented