INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 11, Number 2, Pages 372–384

RECENT ADVANCES ON EXPLICIT VARIATIONAL MULTISCALE A POSTERIORI ERROR ESTIMATION FOR SYSTEMS

GUILLERMO HAUKE, DIEGO IRISARRI, AND FERNANDO LIZARRAGA

(Communicated by R. Celorrio)

This paper is dedicated to F. Lisbona

Abstract. In 1995 the genesis of stabilized methods was established by Professor Hughes from the standpoint of the variational multiscale theory (VMS). By splitting the solution into *resolved* and *unresolved* scales, it was unveiled that stabilized methods take into account an approximation of the *unresolved* scales or *error* into the finite element solution. In this work, the VMS theory is exploited to formulate an explicit a-posteriori error estimator, consistent with the assumptions inherent to stabilized methods.The proposed technology, which is especially suited for fluid flow problems, is very economical and can be implemented in standard finite element codes. It has been shown that, in practice, the method is robust uniformly from the diffusive to the hyperbolic limit.The success of the method can be explained by the fact that in stabilized methods the element local problems for the fine-scale Green's function capture most of the error and the error intrinsic time-scales are an approximation to the solution of the dual problem. Applications to the Euler and linear elasticity equations are shown.

Key words. A posteriori error estimation, stabilized methods.

1. Introduction

The main objective of numerical methods is to obtain reliable approximate solutions. One way of achieving this goal is by quantifying the error and generating adaptive meshes which distribute the error of the numerical solution within the problem domain [1, 2]. This paper summarizes current research on *explicit* a posteriori error estimation for stabilized methods based on the variational multiscale theory (VMS) [25, 27]. This theory is especially suited for estabilized methods and fluid mechanics problems, but it also may find application in solid mechanics.

Within the VMS framework, the first explicit a posteriori error estimator was proposed for the transport equation in [17]. This formulation is a residual-based error estimator and, therefore, the error in each element is estimated as a function of the residual inside the element. There, the capabilities to generate adapted meshes were shown. The resulting method fits in the framework of residual-based methods proposed in [31, 32] but, here, the constants of the error estimates, which are dimensionally consistent, are explicitly given by the theory.

Further achievements on the technology for the transport equation were presented in [13, 17, 18]. Later, the error estimator was extended for the multi-dimensional transport equation in [20], where the jump of the flux along the element edges must be taken into account to attain reliable error estimates in the diffusive dominated

Received by the editors November 1, 2012 and, in revised form, April 11, 2013.
2000 *Mathematics Subject Classification.* 35Q31.
This research was partly supported by the Ministry of Science and Technology of Spain under grant number MTM2010-20463, Spanish MECD under FPU grant (AP210-2073), Departamento de Industria e Innovación del Gobierno de Aragón and the European Social Fund.

regime. These estimators have been tested with practical cases attaining reliable and robust results. All these findings are summarized in [15].

Subsequently, the a posteriori error estimator was extended to the topic of quantities of interest in [19] and to higher-order finite elements [28, 16].

For elliptic problems other methods based on the VMS theory are those of [7, 8, 30, 29]. But in these, the subscales are computed at the element level with the corresponding partial differential equations.

The next challenge consists of extending the present technology to systems of equations. Thus, this paper presents recent advances in relation to the Euler equation and linear elasticity. The error estimator formulation and practical examples are explained in this paper.

2. VMS theory. Error estimation

2.1. The abstract problem. Let Ω be a spatial domain with boundary Γ. The boundary is partitioned into two non-overlapping zones Γ_g and Γ_h such that $\Gamma_g \cup \Gamma_h = \Gamma$ and $\Gamma_g \cap \Gamma_h = \oslash$. The essential boundary condition g is applied on Γ_g and the natural boundary condition, h, on Γ_h.

The strong form of the boundary-value problem consists of finding $u : \Omega \to \mathbb{R}$ such that for the given functions $f : \Omega \to \mathbb{R}$, $g : \Gamma_g \to \mathbb{R}$, $h : \Gamma_h \to \mathbb{R}$, the following equations are satisfied

$$\left\{ \begin{array}{rcllll} \mathcal{L}u & = & f & & \text{in} & \Omega \\ u & = & g & & \text{on} & \Gamma_g \\ \mathcal{B}u & = & h & & \text{on} & \Gamma_h \end{array} \right. \tag{1}$$

with $\mathcal{L}$ being a general differential operator and $\mathcal{B}$, an operator acting on the boundary emanating from integration-by-parts.

In order to introduce the weak form, we have to define suitable spaces for the trial solution, $\mathcal{S}$, and the weighting functions $\mathcal{V}$. The weak form is obtained by multiplying the strong form equation by a weighting function, w, and integrating by parts. Hence, the weak form can be formulated as:

Find $u \in \mathcal{S}$ such that

$$a(w, u) = (w, f) + (w, h)_{\Gamma_h} \qquad \forall w \in \mathcal{V} \tag{2}$$

where $a(\cdot, \cdot)$ is the corresponding bilinear form; $(\cdot, \cdot)$ the $L_2(\Omega)$ inner product and $(\cdot, \cdot)_{\Gamma_h}$, the $L_2(\Gamma_h)$ inner product on Γ_h.

Application of the finite element method necessitates the discretization of the domain Ω into n_{el} non-overlapping elements with domain Ω^e and boundary Γ^e. Let $\widetilde{\Omega}$ and $\widetilde{\Gamma}$ denote the union of element interiors and the inter-element boundaries, respectively,

$$\begin{aligned} \widetilde{\Omega} &= \bigcup_{e=1}^{n_{\text{el}}} \Omega^e \\ \widetilde{\Gamma} &= \bigcup_{e=1}^{n_{\text{el}}} \Gamma^e \backslash \Gamma \end{aligned} \tag{3}$$

In addition, let $[\![\cdot]\!]$ be the jump operator of a function across a discontinuity, for example, an inter-element boundary. According to Fig. 1, the jump of a function