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ON A NUMERICAL TECHNIQUE TO STUDY DIFFERENCE SCHEMES FOR SINGULARLY PERTURBED PARABOLIC REACTION-DIFFUSION EQUATIONS

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This paper is dedicated to the 65th birthday of Professor Francisco J. Lisbona

Abstract. A new technique to study special difference schemes numerically for a Dirichlet problem on a rectangular domain (in x, t) is considered for a singularly perturbed parabolic reaction-diffusion equation with a perturbation parameter ε; $\varepsilon \in (0, 1]$. A well known difference scheme on a piecewise-uniform grid is used to solve the problem. Such a scheme converges ε-uniformly in the maximum norm at the rate $\mathcal{O}\big(N^{-2} \ln^2 N + N_0^{-1}\big)$ as N, $N_0 \to \infty$, where $N + 1$ and $N_0 + 1$ are the numbers of nodes in the spatial and time meshes, respectively; for $\varepsilon \geq m \ln^{-1} N$ the scheme converges at the rate $\mathcal{O}\big(N^{-2} + N_0^{-1}\big)$. In this paper we elaborate a new approach based on the consideration of **regularized errors** in discrete solutions, i.e., **total errors** (with respect to both variables x and t), and also **fractional errors** (in x and in t) generated in the approximation of differential derivatives by grid derivatives. The regularized total errors agree well with known theoretical estimates for actual errors and their convergence rate orders. It is also shown that a "standard" approach based on the "fine grid technique" turns out inefficient for numerical study of difference schemes because this technique brings to large errors already when estimating the total actual error.

Key words. parabolic reaction-diffusion equation, perturbation parameter, boundary layer, difference scheme, piecewise-uniform grids, ε-uniform convergence, numerical experiments, total error, fractional errors, regularized errors.

1. Introduction

At present, a series of theoretically justified numerical methods convergent ε-uniformly has been elaborated for representative classes of singularly perturbed problems (see, e.g., [3, 5] and the bibliography therein). We also know some "heuristic approaches" (they are widely used when solving applied problems) whose justification is rather problematic (see, e.g., [1, 7]). At the same time, nobody knows good experimental methods to study the efficiency of available special grid methods (both theoretical and "heuristic" ones).

Thus, the development of experimental methods for numerical study that allow us to reveal the quality of special schemes is an actual problem in the construction of reliable ε-uniformly convergent grid methods for wide classes of singularly perturbed problems.

Here we could mention only some interesting numerical researches in [1, 7]. In [1], the two-mesh difference technique has been considered for numerical study of difference schemes on piecewise-uniform grids. This technique has been applied, in particular, to solve a two-dimensional elliptic equation on a rectangle in the case when the convective term includes the derivative along the horizontal axis. The schemes have been considered on meshes with the same number of nodes in both

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variables, and this did not allow to reveal the distinct character in the behavior of the errors in different directions. Schemes for singularly perturbed parabolic equations were not considered in [1]. In [2], a high-order scheme in time based on the defect correction technique has been constructed for a parabolic convection-diffusion equation. In the numerical study of this scheme, in order to compute the "exact solution", the main term of the singular solution component (two first terms) written in explicit form was used, and the remainder (smooth part) of the solution was approximated by the corresponding scheme. This approach allowed us to find the errors in x and t (fractional errors). However, even in the case of the *reaction*-diffusion equation such an approach to investigate the errors turns out inapplicable because it is very difficult to write out the main term of the singular component in explicit form.

In the present paper, a parabolic reaction-diffusion equation is considered for which *the convergence rates* of a scheme on a piecewise-uniform grid *are essentially different* for each of variables x and t. The difference scheme converges with the second order up to a logarithmic factor in the spatial variable and with the first order in the time variable that significantly complicates the numerical analysis of the scheme. To study solutions of such discrete problems, a new technique is needed which could allow us to distinguish the behaviour of the errors in each of the variables x and t, i.e., to reveal the fractional errors.

In the present paper, we propose a new approach based on the consideration of *regularized errors* for the discrete solutions, i.e., *total errors* (with respect to both variables x and t), and also *fractional errors* (in x and in t) generated in the approximation of differential derivatives by difference derivatives. The new approach allows us to study effectively special difference schemes on piecewise-uniform grids.

Contents of the paper. The formulation of the initial-boundary value problem for a singularly perturbed parabolic reaction-diffusion equation and the aim of the research are presented in Section 2. Standard difference schemes on uniform and piecewise-uniform grids and estimates of the convergence rates for these schemes are given in Section 3. Some definitions and notations for standard errors (total and fractional errors in space and in time) are introduced in Section 4; the corresponding numerical experiments are discussed in Section 5. Regularized fractional and total errors are introduced in Section 6. Technics for computing the regularized errors and their convergence orders and the corresponding numerical experiments are also considered in Section 6. Conclusions are given in Section 7.

2. Problem formulation and aim of research

On the set $\overline{G}$ with the boundary S

$$\overline{G} = G \cup S, \quad G = D \times (0, T], \quad D = (0, d), \tag{2.1}$$

we consider the initial-boundary value problem for a singularly perturbed parabolic reaction-diffusion equation

$$L\,u(x,t) = f(x,t), \quad (x,t) \in G, \qquad u(x,t) = \varphi(x,t), \quad (x,t) \in S. \tag{2.2}$$

Here* $L_{(2.2)} \equiv \varepsilon^2\, a(x,t)\,\dfrac{\partial^2}{\partial x^2} - c(x,t) - p(x,t)\,\dfrac{\partial}{\partial t}$, $(x,t) \in G$, the functions $a(x,t)$, $c(x,t)$, $p(x,t)$, $f(x,t)$ and $\varphi(x,t)$ are assumed to be sufficiently smooth on $\overline{G}$ and

* Notation $L_{(j.k)}$ $(\overline{G}_{(j.k)}, M_{(j.k)})$ means that this operator (domain, constant) is introduced in the formula $(j.k)$.