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SYMPLECTIC SCHEMES FOR STOCHASTIC HAMILTONIAN SYSTEMS PRESERVING HAMILTONIAN FUNCTIONS

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Abstract. We present high-order symplectic schemes for stochastic Hamiltonian systems preserving Hamiltonian functions. The approach is based on the generating function method, and we prove that the coefficients of the generating function are invariant under permutations for this class of systems. As a consequence, the proposed high-order symplectic weak and strong schemes are computationally efficient because they require less stochastic multiple integrals than the Taylor expansion schemes with the same order.

Key words. Stochastic Hamiltonian systems, generating function, symplectic method, high-order schemes.

1. Introduction

Consider the autonomous stochastic differential equations (SDEs) in the sense of Stratonovich:

$$\begin{aligned} dP_i &= -\frac{\partial H_0(P,Q)}{\partial Q_i}dt - \sum_{r=1}^{m}\frac{\partial H_r(P,Q)}{\partial Q_i}\circ dw_t^r, \quad P(t_0)=p \\ dQ_i &= \frac{\partial H_0(P,Q)}{\partial P_i}dt + \sum_{r=1}^{m}\frac{\partial H_r(P,Q)}{\partial P_i}\circ dw_t^r, \quad Q(t_0)=q, \end{aligned} \tag{1}$$

where P, Q, p, q are n-dimensional vectors with the components $P_i, Q_i, p_i, q_i, i = 1, \dots, n$ and $w_t^r, r = 1, \dots, m$ are independent standard Wiener processes. The SDEs (1) are called the Stochastic Hamiltonian System (SHS) ([12], [11]).

Unlike the deterministic cases, in general the SHS (1) no longer preserves the Hamiltonian functions $H_i, i = 0, \dots, n$ with respect to time. However, by the chain rule of the Stratonovich stochastic integration, for any $i = 0, \dots, m$, we have

$$\begin{aligned} dH_i &= \sum_{k=1}^{n}(\frac{\partial H_i}{\partial P_k}dP_k + \frac{\partial H_i}{\partial Q_k}dQ_k) \\ &= \sum_{k=1}^{n}(-\frac{\partial H_i}{\partial P_k}\frac{\partial H_0}{\partial Q_k} + \frac{\partial H_i}{\partial Q_k}\frac{\partial H_0}{\partial P_k})dt + \sum_{r=1}^{m}\sum_{k=1}^{n}(-\frac{\partial H_i}{\partial P_k}\frac{\partial H_r}{\partial Q_k} + \frac{\partial H_i}{\partial Q_k}\frac{\partial H_r}{\partial P_k})\circ dw_t^r \end{aligned} \tag{2}$$

Thus, the Hamiltonian functions $H_i, i = 0, \dots, m$ are invariant for the flow of the system (1) (i.e. $dH_i = 0$), if and only if $\{H_i, H_j\} = 0$ for $i, j = 0, \dots, m$, where the Poisson bracket is defined as $\{H_i, H_j\} = \sum_{k=1}^{n}(\frac{\partial H_j}{\partial Q_k}\frac{\partial H_i}{\partial P_k} - \frac{\partial H_i}{\partial Q_k}\frac{\partial H_j}{\partial P_k})$. In this paper we propose symplectic schemes for SHS preserving the Hamiltonian functions. This type of SHS is a special case of integrable stochastic Hamiltonian dynamical systems which has been studied in [8]. An example of a SHS preserving Hamiltonian functions is the Kubo oscillator which is used, for instance, in [11] to illustrate the superior performance of symplectic schemes.

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We consider the differential 2-form

$$dp \wedge dq = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n. \tag{3}$$

The stochastic flow $(p, q) \longrightarrow (P, Q)$ of the SHS (1) preserves the symplectic structure (see Theorem 2.1 in [12]) as follows:

$$dP \wedge dQ = dp \wedge dq, \tag{4}$$

i.e. the sum of the oriented areas of projections of a two-dimensional surface onto the coordinate planes (p_i, q_i), $i = 1, \ldots, n$, is invariant. It should be noted that in (1) p, q are fixed parameters and the differentiation is done with respect to time t, while in (4) the differentiation is carried out with respect to the initial data p, q. We say that a method based on the one-step approximation $\bar{P} = \bar{P}(t+h; t, p, q)$, $\bar{Q} = \bar{Q}(t+h; t, p, q)$ preserves symplectic structure if

$$d\bar{P} \wedge d\bar{Q} = dp \wedge dq. \tag{5}$$

If for the approximation $\bar{X}_k = (\bar{P}, \bar{Q})$, $k = 0, 1, \ldots$, of the solution $X(t_k, \omega) = (P(t_k, \omega), Q(t_k, \omega))$, we have

$$\left[E|\bar{X}_k(\omega)) - X(t_k, \omega)|^2\right]^{1/2} \leq Kh^j, \tag{6}$$

where $t_k = t_0 + kh \in [t_0, t_0 + T]$, h is the time step, and the constant K does not depend on k and h, then we say that $\bar{X}_k$ approximates the solution $X(t_k)$ of (1) with mean square order of accuracy j ([7]). On the other hand, if

$$|E[F(\bar{X}_k(\omega))] - E[F(X(t_k, \omega)]| \leq Kh^j, \tag{7}$$

for F from a sufficiently large class of functions, where $t_k = t_0 + kh \in [t_0, t_0 + T]$, h is the time step, and the constant K does not depend on k and h, then $\bar{X}_k$ approximates the solution $X(t_k)$ of (1) in the weak sense with weak order of accuracy j ([7]).

Milstein et al. [12] [11] have constructed a symplectic scheme with mean square order 0.5 for the general SHS (1), and several symplectic schemes with higher mean square order for special types of SHSs such as SHSs with additive noise or separable Hamiltonians. A symplectic scheme with weak order one is constructed in [10]. An approach to construct symplectic schemes for SHSs based on generating functions was proposed by Wang in [13]. More recently, Wang et al. have also proposed variational integrators [14] for SHSs, and have presented several applications of the generating functions method for SHSs in [5], [6].

In [1] and [4], we follow the approach based on generating functions and we propose a recurrence formula for finding the coefficients of the generating function for SHSs. We derive several higher order strong and weak schemes and we also illustrate by numerical simulations that symplectic schemes are more accurate for long term numerical calculations than the non-symplectic methods. In this study, we extend the results presented in [3] and we focus on SHS preserving the Hamiltonian functions.

In the next section, we introduce general results regarding the generating functions associated with the SHS (1). The main results are presented in Section 3 where we prove that the coefficients of the generating function are invariant under permutation for this type of systems. Hence, the construction of the strong and weak symplectic schemes of order two and three reported in Section 4 is simpler and more efficient than the non-symplectic explicit Taylor expansion schemes with the same order. In Section 5 we illustrate numerically the performance of the proposed strong and weak symplectic schemes.