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GLOBAL H^2-REGULARITY RESULTS OF THE 3D PRIMITIVE EQUATIONS OF THE OCEAN

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Abstract. In this article, we consider the 3D viscous primitive equations (PEs for brevity) of the ocean under two physically relevant boundary conditions for the H^1 and H^2 smooth initial data, respectively. The H^2 regularity result of the solution for the viscous PEs of the ocean has been unknown since the work by Cao and Titi [3], and Kobelkov [26]. In this article we provide the global H^2-regularity results of the solution and its time derivatives for the 3D viscous primitive equations of the ocean by using the L^6 estimates developed in [3] and some new energy estimate techniques.

Key words. Primitive equations, ocean, regularity.

1. Introduction

Given a smooth bounded domain $\omega \subset \mathbb{R}^2$ and the cylindrical domain $\Omega = \omega \times (-d, 0) \subset \mathbb{R}^3$, we consider in Ω the following 3D viscous PEs of the ocean with rigid lid approximation and in the presence of one stratification:

$$u_t + L_1 u + (u \cdot \nabla)u + w\partial_z u + \nabla P + f\vec{k} \times u = F_1, \tag{1.1}$$

$$\theta_t + L_2\theta + (u \cdot \nabla)\theta + w\partial_z\theta - \sigma w = F_2 \tag{1.2}$$

$$\nabla \cdot u + \partial_z w = 0, \tag{1.3}$$

$$\partial_z P + \gamma\theta = 0. \tag{1.4}$$

The unknowns for the 3D viscous PEs are the fluid velocity field $(u, w) = (u_1, u_2, w) \in \mathbb{R}^3$ with $u = (u_1, u_2)$ being the horizontal velocity, the density θ and the pressure P. Here $f = f_0(\beta + y)$ is the given Coriolis rotation frequency with β-plane approximation, F_1 and F_2 are two given functions and $\vec{k}$ is the vertical unit vector, $\sigma > 0$ is the stratification constant of the ocean and $\gamma > 0$ is the gravitational constant. The elliptic operators L_1 and L_2 are given respectively as the following:

$$L_i = -\nu_i\Delta - \mu_i\partial_z^2, \ i = 1, \ 2.$$

Here the positive constants $\nu_1, \ \mu_1$ are the horizontal and vertical viscosity coefficients; while the positive constants $\nu_2, \ \mu_2$ are the horizontal and vertical thermal diffusivity coefficients and

$$\nabla = (\partial_x, \partial_y), \ \Delta = \partial_{xx} + \partial_{yy}, \ \partial_{x_i} = \frac{\partial}{\partial x_i}, \ \partial_{x_i x_i} = \partial_{x_i}^2,$$

with $i = 1, 2, 3$ and $(x_1, x_2, x_3) = (x, y, z)$.

For more details on the PEs of the ocean, the reader is referred to [7, 28, 29, 34] for the physical aspect, and to [3, 20, 25, 22, 23, 24, 31, 32, 16, 17, 18, 19, 25] for the mathematical aspect.

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Here and after, we use the following notations:

$$\eta_t = \frac{\partial \eta}{\partial t},\ \phi_{x_i} = \partial_{x_i}\phi,\ \psi_{x_ix_i} = \partial_{x_ix_i}\psi,$$

with $i = 1, 2, 3$ for any $\eta(t) \in H^1(0,\infty)$, $\phi(x,y,z) \in H^1(\Omega)$ and $\psi(x,y,z) \in H^2(\Omega)$.

We partition the boundary of Ω into the following three parts:

$$\Gamma_u = \{(x,y,z) \in \bar{\Omega}; z = 0\},$$
$$\Gamma_b = \{(x,y,z) \in \bar{\Omega}; z = -d\},$$
$$\Gamma_s = \{(x,y,z) \in \bar{\Omega}; (x,y) \in \partial\omega,\ -d \le z \le 0\}.$$

Next, we provide the system (1.1)-(1.4) with the following boundary conditions-with the wind-driven on the top surface and non-slip and non-heat flux on the side walls and the bottom (see, e.g., page 246 in [3], page 160 in [20] and page 1037 in [25]):

$$\begin{aligned}
&\text{on } \Gamma_u,\ \partial_z u = d\,\tau^*,\ w = 0,\ \partial_z\theta = -\alpha(\theta - \theta^*);\\
&\text{on}\Gamma_b,\ \partial_z u = 0,\ w = 0, \partial_z\theta = 0;\\
&\text{on } \Gamma_s,\ u\cdot\mathbf{n} = 0,\ \frac{\partial u}{\partial \mathbf{n}} \times \mathbf{n} = \mathbf{0},\ \frac{\partial\theta}{\partial\mathbf{n}} = \mathbf{0},\\
&\text{or on } \Gamma_s,\ u = 0,\ \frac{\partial\theta}{\partial\mathbf{n}} = 0,
\end{aligned}$$

where $\tau^* = \tau^*(x,y)$ is the wind stress on the ocean surface, α is a positive constant, $\mathbf{n}$ is the normal vector of Γ_s and $\theta^* = \theta^*(x,y)$ is the typical density distribution of the top surface of the ocean. Based on the above condition, it is natural to assume that $\tau^*(x,y)$ and $\theta^*(x,y)$ satisfy

$$\tau^*\cdot\mathbf{n} = \mathbf{0},\ \frac{\partial\tau^*}{\partial\mathbf{n}} \times \mathbf{n} = \mathbf{0},\quad \text{or } \tau^* = \mathbf{0},\ \text{and } \frac{\partial\theta^*}{\partial\mathbf{n}} = \mathbf{0} \text{ on } \partial\omega.$$

Due to this condition, we can convert the previous boundary condition into the homogeneous by replacing (u,θ) by $(u + \frac{1}{2}([(z+d)^2 - \frac{1}{3}d^3]\tau^*, \theta + \theta^*)$ (refer to page 248 in [3]).

Hence, we consider the following boundary conditions for the 3D viscous PEs:

$$w|_{\Gamma_u\cup\Gamma_b} = 0. \tag{1.5}$$

$$\partial_z u|_{\Gamma_u\cup\Gamma_b} = 0,\ u\cdot\mathbf{n}|_{\Gamma_s} = 0,\ \frac{\partial u}{\partial\mathbf{n}} \times \mathbf{n}|_{\mathbf{\Gamma_s}} = \mathbf{0}; \tag{1.6-1}$$

$$\text{or } \partial_z u|_{\Gamma_u\cup\Gamma_b} = 0,\ u|_{\Gamma_s} = 0; \tag{1.6-2}$$

$$\partial_z\theta|_{\Gamma_b} = (\partial_z\theta + \alpha\theta)|_{\Gamma_u} = 0,\ \frac{\partial\theta}{\partial\mathbf{n}}|_{\Gamma_s} = 0, \tag{1.7}$$

refer to (28)-(29) of page 248 in [3] and (1.3)-(1.4) of page 160 in [20] for the boundary condition (1.5), (1.6-1) and (1.7), and Remark 2.1 of page 1038 in [25] for the boundary condition (1.5), (1.6-2) and (1.7).

Also, the initial conditions of $u(x,y,z,t)$ and $\theta(x,y,z,t)$ should be given by

$$u(x,y,z,0) = u_0(x,y,z),\ \theta(x,y,z,0) = \theta_0(x,y,z). \tag{1.8}$$

Using the Dirichlet boundary condition (1.5) of w on $\Gamma_u \cap \Gamma_b$ and (1.3)-(1.4), we have

$$w(x,y,z,t) = -\int_{-d}^{z} \nabla\cdot u(x,y,\xi,t)d\xi,\ \int_{-d}^{0} \nabla\cdot u(x,y,\xi,t)d\xi = 0,$$
$$P(x,y,z,t) = p(x,y,t) - \gamma\int_{-d}^{z} \theta(x,y,\xi,t)d\xi.$$