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ON LOCAL SUPER-PENALIZATION OF INTERIOR PENALTY DISCONTINUOUS GALERKIN METHODS

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Abstract. We prove in an abstract setting that standard (continuous) Galerkin finite element approximations are the limit of interior penalty discontinuous Galerkin approximations as the penalty parameter tends to infinity. We apply this result to equations of non-negative characteristic form and the non-linear, time dependent system of incompressible miscible displacement. Moreover, we investigate varying the penalty parameter on only a subset of a triangulation and the effects of local super-penalization on the stability of the method, resulting in a partly continuous, partly discontinuous method in the limit. An iterative automatic procedure is also proposed for the determination of the continuous region of the domain without loss of stability of the method.

Key words. discontinuous Galerkin methods, finite elements, interior penalty.

1. Introduction

The discontinuous Galerkin (dG) finite element method has become widely used in recent years for a variety of problems as it possesses several desirable qualities, such as: good stability properties due to the natural incorporation of upwinding techniques; flexible mesh design as hanging nodes and irregular meshes are admissible; and relatively easy implementation of *hp*-adaptive algorithms. These properties however come with the drawback of an increased number of degrees of freedom compared to a standard conforming method. For instance, when using an axi-parallel quadrilateral mesh in two dimensions with piecewise bilinear elements for which the standard continuous Galerkin (cG) finite element method has approximately n degrees of freedom (depending on boundary conditions) the dG method on the same mesh has approximately $4n$ degrees of freedom.

For advection-dominated advection-diffusion-reaction equations the standard cG method exhibits poor stability properties and non physical oscillations may pollute the approximation globally. Discontinuous Galerkin methods have generally better stability properties. In the case of interior penalty dG method, for instance, stability in the upwind direction has been shown in the inf-sup sense, e.g., in [1, 8], generalizing ideas from [21], where purely hyperbolic problems were considered.

Conceptually, somewhere between the standard cG and interior penalty dG methods lies the continuous-discontinuous Galerkin (cdG) finite element method [12], whereby one seeks a Galerkin solution on a finite element space V_{cdG} with $V_{\mathrm{cG}} \subset V_{\mathrm{cdG}} \subset V_{\mathrm{dG}}$, where V_{cG} and V_{dG} are the standard cG and dG finite element spaces. In the context of problems with layers or sharp fronts, continuous elements can be used away from the layers/fronts and discontinuous elements (accommodating appropriate upwinding) can be used in the region where the layers/fronts are present. This idea has been studied previously in the context of problems with layers by Dawson and Proft [16] using transmission conditions between regions where

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different spaces are used. Cangiani, Georgoulis and Jensen [12] and Devloo, Forti and Gomez [17] have previously compared the cdG finite element method with alternative methods for advection-diffusion equations.

The control of discontinuities across element interfaces in the dG framework can be exercised by introducing and/or tuning the, so-called, jump penalization parameters. Using excessive penalization within a dG approximation will be referred to as the *super penalty method*. It is natural to expect that as the penalty parameter is increased the interelement jumps in the numerical approximation decrease. It has been shown by Larson and Niklasson [22] for stationary linear elliptic problems (using the interior penalty method) and by Burman, Quarteroni and Stamm [9] for stationary hyperbolic problems (penalising the jumps of the approximation for discontinuous elements and the jumps in the gradient of the approximation for continuous elements) that the dG approximation converges to the cG approximation as the jump penalization parameter tends to infinity.

In this work, our aim is twofold. Firstly, we present an alternative proof of the convergence of dG methods to cG methods, using a far more general framework covering the cases considered by [9, 22] and also non-linear and time dependent problems. Moreover, we show that super-penalization procedures can be localized to designated element faces, thereby arriving to partly continuous, partly discontinuous finite element methods. As particular examples we consider the limits of the interior penalty dG method for PDEs with non-negative characteristic form [20] and the mixed Raviart-Thomas-dG method for the miscible displacement system presented in [4].

Secondly, we continue the numerical investigations of [12] in the context of blending locally continuous and discontinuous methods. In particular, we investigate to what degree numerical oscillations appear as local super-penalization is applied. The aim, of course, is to find the extent to which degrees of freedom can be removed by using locally continuous finite element spaces without affecting the extra stability offered by dG methods. To this end, we consider an advection-dominated advection-diffusion problem containing boundary layer behaviour, where the continuous and the discontinuous regions of the finite element solution are tuned manually. A second example investigates the use of an iterative automatic procedure for the determination of the continuous region of the domain by local super-penalization without loss of stability of the method. The procedure is applied to the problem of incompressible miscible displacement.

This work is organized as followed. After introducing notation in Section 1.1 an abstract discussion of the limit of penalty methods is given in Section 2. We then show how this framework can be applied to equations of non-negative characteristic form in Section 3 and to the non-linear equations of incompressible miscible displacement in Section 4. Finally, Section 5 contains a number of numerical experiments and discussion of an iterative automatic procedure for determining the continuous regions of the approximation.

1.1. Notation. Let $\Omega \subset \mathbb{R}^d$ be a bounded, open polygonal domain. We denote by $\mathcal{T}_h$ a subdivision of Ω into open non-overlapping d-simplices E. The diameter of $E \in \mathcal{T}_h$ is denoted by h_E. Let also $\mathcal{E}_h := \cup_{E\in\mathcal{T}_h}\partial E$ be the skeleton of the mesh $\mathcal{T}_h$, while $\mathcal{E}_h^o := \mathcal{E}_h \backslash \partial\Omega$. Finally, let Γ denote the set of elemental boundary faces, i.e., those which lie in $\partial\Omega$.

For $e \in \mathcal{E}_h^o$, with $e = \bar{E}^+ \cap \bar{E}^-$ for E^+, $E^- \in \mathcal{T}_h$, we define $h_e := \min(h_{E^-}, h_{E^+})$. Given a generic scalar field $\nu : \Omega \to \mathbb{R}$ that may be discontinuous across e, we set $\nu^\pm := \nu|_{E^\pm}$, the interior trace on $E^\pm$. Similarly, for a generic vector field