INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 11, Number 3, Pages 496–524

TWO-GRID *hp*-VERSION DISCONTINUOUS GALERKIN FINITE ELEMENT METHODS FOR QUASI-NEWTONIAN FLUID FLOWS

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Abstract. In this article we consider the *a priori* and *a posteriori* error analysis of two-grid *hp*-version discontinuous Galerkin finite element methods for the numerical solution of a strongly monotone quasi-Newtonian fluid flow problem. The basis of the two-grid method is to first solve the underlying nonlinear problem on a coarse finite element space; a fine grid solution is then computed based on undertaking a suitable linearization of the discrete problem. Here, we study two alternative linearization techniques: the first approach involves evaluating the nonlinear viscosity coefficient using the coarse grid solution, while the second method utilizes an incomplete Newton iteration technique. Energy norm error bounds are deduced for both approaches. Moreover, we design an *hp*-adaptive refinement strategy in order to automatically design the underlying coarse and fine finite element spaces. Numerical experiments are presented which demonstrate the practical performance of both two–grid discontinuous Galerkin methods.

Key words. *hp*-finite element methods; discontinuous Galerkin methods, a posteriori error estimation, adaptivity, two-grid methods, non-Newtonian fluids

1. Introduction

The purpose of this article is to develop the *a priori* and *a posteriori* error analysis of two-grid *hp*-version discontinuous Galerkin finite element methods (DGFEMs) for the numerical approximation of strongly monotone quasi-Newtonian fluid flow problems. The general philosophy of two-grid methods is to first approximate the underlying nonlinear problem on a coarse finite element partition of the computational domain. On the basis of this coarse grid approximation, a linearized variant of the discrete problem is then computed on a fine mesh; see, for example, [4, 8, 9, 16, 23, 27, 30, 31, 32, 33], and the references cited therein. The linearization required for the construction of the linear problem to be solved on the fine mesh may be undertaken in a number of different ways. Indeed, a simple approach is to evaluate the nonlinear coefficients arising in the underlying partial differential equation using the coarse grid approximation; in this way, the fine grid approximation essentially includes an underlying modelling or data approximation error stemming from *fixing* the data of the problem, cf. [9, 33], for example. On the other hand, the coarse grid solution may be used as the initial guess for a Newton solver on the fine mesh; in this setting, one step of the Newton iteration technique is typically undertaken, cf. [4, 33]. In the context of DGFEMs, Bi & Ginting [9] studied the first approach for a class of quasilinear elliptic PDEs, where the nonlinear diffusion coefficient μ depends on the analytical solution u; this analysis was then extended to consider the case when $\mu = \mu(|\nabla u|)$ in [15]. The analysis of DGFEMs using the two-grid technique based on employing a single step of a Newton solver for this latter class of scalar PDEs has been studied in [12].

In this article we generalize the analysis presented in [12, 14, 15] for two-grid *hp*-version interior penalty (IP) DGFEM approximations of scalar quasilinear elliptic

Received by the editors October 02, 2013.
2000 *Mathematics Subject Classification.* 65N30, 65N55, 65M60.

PDEs to the following non-Newtonian fluid flow problem:

$$-\nabla \cdot \{\mu\left(\boldsymbol{x}, |\underline{e}\left(\boldsymbol{u}\right)|\right) \underline{e}\left(\boldsymbol{u}\right)\} + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{2}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma. \tag{3}$$

Here, $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded polygonal, or polyhedral, Lipschitz domain with boundary $\Gamma = \partial\Omega$, $\boldsymbol{f} \in L^2(\Omega)^d$ is a given source term, $\boldsymbol{u} = (u_1, \ldots, u_d)^\top$ is the velocity vector, p is the pressure, and $\underline{e}(\boldsymbol{u})$ is the symmetric $d \times d$ strain tensor defined by

$$e_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad i, j = 1 \ldots d.$$

Furthermore, $|\underline{e}(\boldsymbol{u})|$ is the Frobenius norm of $\underline{e}(\boldsymbol{u})$. For the purposes of this article we assume that the function μ satisfies the following structural hypothesis.

Assumption 1. We assume that the nonlinearity $\mu \in \mathrm{C}(\bar{\Omega} \times [0, \infty))$ and there exists positive constants m_μ and M_μ such that

$$m_\mu(t - s) \leq \mu(\boldsymbol{x}, t)t - \mu(\boldsymbol{x}, s)s \leq M_\mu(t - s), \qquad t \geq s \geq 0, \quad \boldsymbol{x} \in \bar{\Omega}. \tag{4}$$

As a direct consequence of (4), we note that μ satisfies the following inequalities: there exists positive constants C_1 and C_2, such that for all $\underline{\tau}, \underline{\omega} \in \mathbb{R}^{d\times d}$ and all $\boldsymbol{x} \in \bar{\Omega}$,

$$|\mu(\boldsymbol{x}, |\underline{\tau}|)\underline{\tau} - \mu(\boldsymbol{x}, |\underline{\omega}|)\underline{\omega}| \leq C_1|\underline{\tau} - \underline{\omega}|, \tag{5}$$

$$C_2|\underline{\tau} - \underline{\omega}|^2 \leq (\mu(\boldsymbol{x}, |\underline{\tau}|)\underline{\tau} - \mu(\boldsymbol{x}, |\underline{\omega}|)\underline{\omega}) : (\underline{\tau} - \underline{\omega}). \tag{6}$$

For ease of notation we suppress the dependence of μ on $\boldsymbol{x}$ and write $\mu(t)$ instead of $\mu(\boldsymbol{x}, t)$. Throughout this paper, we use the following standard notation. Given $D \subset \mathbb{R}^d, d \geq 1$, a bounded Lipschitz domain, we write $H^t(D)$ to denote the usual Sobolev space of real-valued functions of order $t \geq 0$ with norm $\|\cdot\|_{H^t(D)}$; for $t = 0$, we set $L^2(D) = H^0(D)$. We write $H_0^1(D)$ to denote the subspace of functions in $H^1(D)$ with zero trace on ∂D, and set $L_0^2(D) = \{q \in L^2(D) : \int_D q \,\mathrm{d}\boldsymbol{x} = 0\}$.

This article is organized as follows. In Section 2 we formulate the standard (single grid) IP DGFEM for the numerical approximation of the boundary-value problem (1)–(3). Sections 3 and 4 develop the *a priori* and *a posteriori* error analysis of the above mentioned variants of the two–grid *hp*-version IP DGFEM. On the basis of the *a posteriori* error bounds established in this article, in Section 5 we consider the design of an *hp*-adaptive finite element algorithm capable of automatically enriching the underlying coarse and fine finite element spaces; the performance of this adaptive strategy is studied in Section 6. Finally, in Section 7 we summarise the main results of this article and draw some conclusions.

2. *hp*-Version IP DGFEM approximation

In order to construct the two-grid IP DGFEMs considered in this article, we first recall the family of IP DGFEMs introduced and analysed in [13]. To this end, we introduce the following notation. Let $\mathcal{T}_h$ denote a shape-regular quadrilateral/hexahedral partition of the computational domain Ω into disjoint open-element domains κ such that $\overline{\Omega} = \bigcup_{\kappa\in\mathcal{T}_h} \overline{\kappa}$. We assume that each $\kappa \in \mathcal{T}_h$ is an affine image of a fixed master element $\widehat{\kappa}$; i.e., for each $\kappa \in \mathcal{T}_h$, there exists an affine mapping $T_\kappa : \widehat{\kappa} \to \kappa$ such that $\kappa = T_\kappa(\widehat{\kappa})$, where $\widehat{\kappa} = (-1, 1)^d$ is the reference element. We write h_κ to denote the element diameter of $\kappa \in \mathcal{T}_h$, and set $h = \max_{\kappa\in\mathcal{T}_h} h_\kappa$; furthermore, $\boldsymbol{n}_\kappa$ signifies the unit outward normal vector to κ. We allow the meshes $\mathcal{T}_h$ to be one-irregular and assume that the family of meshes $\{\mathcal{T}_h\}_{h>0}$ is of 'bounded