INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 11, Number 3, Pages 567–586

ENERGY NORM ERROR ESTIMATES FOR AVERAGED DISCONTINUOUS GALERKIN METHODS IN 1 DIMENSION

GÁBOR CSÖRGŐ AND FERENC IZSÁK

Abstract. Numerical solution of one-dimensional elliptic problems is investigated using an averaged discontinuous discretization. The corresponding numerical method can be performed using the favorable properties of the discontinuous Galerkin (dG) approach, while for the average an error estimation is obtained in the H^1-seminorm. We point out that this average can be regarded as a lower order modification of the average of a well-known overpenalized symmetric interior penalty (IP) method. This allows a natural derivation of the overpenalized IP methods.

Key words. discontinuous Galerkin method, smoothing technique, and error estimation.

1. Introduction

Discontinuous Galerkin (dG) methods have been intensively studied in the last decade. Due to the increasing need for highly accurate computation, these methods, allowing local refinement strategies, became very popular. Their unified mathematical analysis for elliptic boundary value problems was initiated in [2], and a number of articles have been published discussing its application to different problems. The theory was put later in a more general framework [10], [11], [12]. Regarding the practical computations, also some monographs have been appeared [15], [21]. The widespread results of the theoretical investigation for dG methods have been summarized recently in [8].

The error analysis for elliptic boundary value problems underwent a significant development. For the multidimensional case, extra smoothness of the analytic solution had been assumed in the original approach [2], which was alleviated in [14]. The a posteriori error analysis was initiated in [18] and [3] and was developed in [1] and [13] to obtain easily computable and guaranteed error bounds and an efficient a posteriori error estimator for a general 3-dimensional *hp*-adaptive algorithm has been derived in [22]. All of these results concern a so-called dG-norm which arises from the dG bilinear form. One can prove convergence also in a mesh-independent (BV) norm [4], [7], which can be used again to avoid the assumption on extra smoothness [8].

Several methods have been developed to obtain an error estimator in the L_2-norm and increase the accuracy of the dG approximation in negative Sobolev norms. The key idea is to apply a post-processing which is a smoothing technique using convolution with special kernels. This was first demonstrated in [6] for hyperbolic problems. These techniques have been developed in many aspects, for recent achievements see, *e.g.*, [17] and [19]. Similar results including superconvergence can be obtained for second-order elliptic problems in several space dimensions [5] using an element-by-element postprocessing in the L_2-norm.

Received by the editors September 3, 2013 and, in revised form, January 19 2014.
2000 *Mathematics Subject Classification.* 65N12, 65N15, 65N30 .
The authors thank the reviewers for their valuable comments. F. Izsák acknowledges the support of the Hungarian Research Fund OTKA (grants PD10441 and K104666).

The objective of this paper is to develop an error estimator *in the natural energy seminorm* between a postprocessed dG type approximation and the analytical solution in one space dimension for elliptic problems. In the main result (see Theorem 3), we provide an upper estimate for the error $\|\nabla(\eta_h * u_{\mathrm{IP}} - u)\|_{L_2}$, where the convolution gives the local average (a kind of postprocessing) and u_{IP} denotes an overpenalized version of the well-known symmetric interior penalty (IP) approximation.

We also throw new light upon a version of dG methods: we will point out that a postprocessed IP method can be regarded as a lower order modification of a continuous Galerkin method. In turn, this suggests a new derivation of a family of overpenalized IP methods, where instead of a heuristic choice the penalty term arises in a natural way.

These results are also confirmed in numerical experiments: the local average of the proposed method and that of the overpenalized IP method are really close to each other. Also, it will be verified that for the local averages the convergence in the H^1 seminorm is valid.

After the preliminaries, we introduce the finite element method which can be recognized both as a continuous and a discontinuous method. Then the corresponding bilinear form is analyzed first in a simple situation and then its relation with the interior penalty method is highlighted. Finally, we prove error estimation between the postprocessed solution and the analytic one. As a consequence, we obtain the above energy norm error estimation for a simple local average of the interior penalty approximation.

2. Mathematical preliminaries

We investigate the finite element solution of the one-dimensional elliptic boundary value problem

$$\begin{cases} -\Delta u = f \quad \text{in } \Omega = (a,b) \subset \mathbb{R} \\ u(a) = u(b) = 0, \end{cases} \tag{1}$$

where $f \in L_2(a,b)$ is given. For the numerical solution we consider a tessellation of the interval (a,b) into the disjoint subintervals $I_0, I_1, \dots, I_n$ such that

$$I_j = (\gamma_j, \gamma_{j+1}), \quad a = \gamma_0 < \gamma_1 < \dots \gamma_{n+1} = b.$$

The parameter h with

$$h = \min_{j \in \{1,2,\dots,n+1\}} (\gamma_j - \gamma_{j-1})$$

denotes the minimal length of the subintervals.

The vector space for the polynomials of maximal degree k on the interval I is denoted with $\mathcal{P}_k(I)$. For $\mathbf{k} = (k_0, k_1, \dots, k_n)$ we define

$$P_{h,\mathbf{k}}(a,b) = \mathcal{P}_{k_0}(I_0) \oplus \mathcal{P}_{k_1}(I_1) \oplus \dots \oplus \mathcal{P}_{k_n}(I_n),$$

the direct sum of the above polynomial spaces which corresponds to the piecewise polynomials with the given maximal degree $k_0, k_1, \dots, k_n$ on $I_0, I_1, \dots, I_n$.

The symbol $(\cdot,\cdot)_{I_*}$ refers to the $L_2(I_*)$ scalar product on $I_* \subset (a,b)$. If $I_* = (a,b)$ we omit the subscript. Accordingly, the generated $L_2(I_*)$-norm is denoted with $\|\cdot\|_{I_*}$, where $I_* = (a,b)$ will be omitted.

For the numerical solution we use the family of the average and jump operators $\{\!\{\cdot\}\!\}_j$ and $[\![\cdot]\!]_j$, which are defined by

$$\{\!\{u\}\!\}_j = \frac{1}{2} \cdot (\lim_{\gamma_j -} u + \lim_{\gamma_j +} u) \quad \text{and} \quad [\![u]\!]_j = \lim_{\gamma_j -} u - \lim_{\gamma_j +} u$$