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TWO-LEVEL PENALTY FINITE ELEMENT METHODS FOR NAVIER-STOKES EQUATIONS WITH NONLINEAR SLIP BOUNDARY CONDITIONS

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Abstract. The two-level penalty finite element methods for Navier-Stokes equations with nonlinear slip boundary conditions are investigated in this paper, whose variational formulation is the Navier-Stokes type variational inequality problem of the second kind. The basic idea is to solve the Navier-Stokes type variational inequality problem on a coarse mesh with mesh size H in combining with solving a Stokes type variational inequality problem for simple iteration or solving a Oseen type variational inequality problem for Oseen iteration on a fine mesh with mesh size h. The error estimate obtained in this paper shows that if $H = O(h^{5/9})$, then the two-level penalty methods have the same convergence orders as the usual one-level penalty finite element method, which is only solving a large Navier-Stokes type variational inequality problem on the fine mesh. Hence, our methods can save a amount of computational work.

Key words. Navier-Stokes Equations; Nonlinear Slip Boundary Conditions; Variational Inequality Problem; Penalty Finite Element Method; Two-Level Methods.

1. Introduction

Constructing efficient algorithms for solving Navier-Stokes equations is a fundamental and important problem. A difficulty lies in that the velocity and the pressure are coupled by the solenoidal condition. The popular technique to overcome this difficulty is relaxing the solenoidal condition in an appropriate method and resulting in a pesudo-compressible system, such as the penalty method introduced by Temam in [1,2], the locally stabilized methods introduced by Kechkar in [3], the pressure projection stabilized methods introduced by Bochev in [4] and Li in [5] and the references cited therein.

The other difficulty is that the Navier-Stokes equations are nonlinear. The two-level method is a very popular technique for solving the numerical solutions of the nonlinear equations. Its main idea is to solve a nonlinear problem on a coarse mesh and solving a linear problem on a fine mesh, which saves computational work for solving a nonlinear problem. There are a large amount of papers about the two-level method, such as for nonlinear partial differential equations [6-11] and especially for Navier-Stokes equations with homogeneous Dirichlet boundary conditions [12-21].

In this paper, we will consider the two-level penalty finite element methods for Navier-Stokes equations with nonlinear slip boundary conditions. Since the nonlinear boundary conditions are from the subdifferential property on the part boundary, the weak variational formulation is the variational inequality problem of the second kind with Navier-Stokes operator which is called the Navier-Stokes type variational inequality problem. This nonlinear slip boundary conditions are firstly

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introduced by Fujita in [22] and appear in the modeling of blood flow in a vein of an arterial sclerosis patient. The approach consists solving the Navier-Stokes type variational inequality problem on a coarse mesh with mesh size H in combining with solving a Stokes type variational inequality problem for simple iteration or solving a Oseen type variational inequality problem for Oseen iteration on a fine mesh with mesh size h. Denote $u^{\varepsilon h}$ and $p^{\varepsilon h}$ the penalty approximation solutions on the fine mesh. The error estimate derived in this paper is

$$||u - u^{\varepsilon h}||_V + ||p - p^{\varepsilon h}|| \le c(\varepsilon + h^{5/4} + H^{9/4}),$$

where $c > 0$ is independent of h and H. This error estimate shows that if $H = O(h^{5/9})$ and ε is sufficiently small, the two-level penalty finite element methods have the same convergence orders as the usual one-level penalty finite element methods studied in [23]. Hence, our methods can save the CPU time and improve the computational efficiency.

2. Navier-Stokes Equations with Nonlinear Slip Boundary Conditions

Let $\psi : \mathbb{R}^2 \to \overline{\mathbb{R}} = (-\infty, +\infty]$ be a given function possessing the properties of convexity and weak semi-continuity from below (ψ is not identical with $+\infty$). The subdifferential set $\partial\psi(a)$ denotes a subdifferential of the function ψ at the point a:

$$\partial\psi(a) = \{b \in \mathbb{R}^2 : \psi(t) - \psi(a) \ge b \cdot (t - a), \quad \forall\, t \in \mathbb{R}^2\}.$$

Consider the steady Navier-Stokes equations

$$\begin{cases} -\mu\Delta u + (u\cdot\nabla)u + \nabla p = f & \text{in } \Omega, \\ \text{div} u = 0 & \text{in } \Omega \end{cases} \tag{1}$$

with the following nonlinear slip boundary conditions [22]:

$$\begin{cases} u = 0, & \text{on } \Gamma, \\ u_n = 0, \quad -\sigma_\tau(u) \in g\partial|u_\tau| & \text{on } S, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^2$, is a bounded convex domain. $\Gamma \cap S = \emptyset, \overline{\Gamma \cup S} = \partial\Omega$. The viscous coefficient $\mu > 0$ is a positive constant. g is the scalar functions; $u_n = u \cdot n$ and $u_\tau = u - u_n n$ are the normal and tangential components of the velocity, where n stands for the unit vector of the external normal to S; $\sigma_\tau(u) = \sigma - \sigma_n n$, independent of p, is the tangential component of the stress vector σ which is defined by $\sigma_i = \sigma_i(u, p) = (\mu e_{ij}(u) - p\delta_{ij})n_j$, where $e_{ij}(u) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}, i, j = 1, 2$. From the definition of the subdifferential property, we note that the variational formulation of (1)-(2) is the variational inequality problem of the second kind with Navier-Stokes operator.

To give the variational formulation, we introduce some spaces which we will need later in this paper. Denote

$$V = \{u \in H^1(\Omega)^2,\ u|_\Gamma = 0,\ u \cdot n|_S = 0\}, \quad V_0 = H_0^1(\Omega)^2,$$

$$V_\sigma = \{u \in V, \quad \text{div} u = 0\}, \qquad M = L_0^2(\Omega) = \{q \in L^2(\Omega), \int_\Omega q \text{d}x = 0\}.$$

Let $|| \cdot ||_k$ be the norm in Hilbert space $H^k(\Omega)^2$. Let $(\cdot,\cdot)$ and $||\cdot||$ be the inner product and the norm in $L^2(\Omega)^2$. Then we can define the inner product and the norm in V by $(\nabla\cdot, \nabla\cdot)$ and $||\cdot||_V = ||\nabla\cdot||$, respectively, because $||\nabla\cdot||$ is equivalent to $||\cdot||_1$. Let $\mathbb{X}$ be a Banach space. Denote $\mathbb{X}'$ the dual space of $\mathbb{X}$ and $< \cdot, \cdot >$ be the dual pairing in $\mathbb{X} \times \mathbb{X}'$.