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ANALYSIS OF AN AUGMENTED FULLY-MIXED FINITE ELEMENT METHOD FOR A THREE-DIMENSIONAL FLUID-SOLID INTERACTION PROBLEM

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Abstract. We introduce and analyze an augmented fully-mixed finite element method for a fluid-solid interaction problem in 3D. The media are governed by the acoustic and elastodynamic equations in time-harmonic regime, and the transmission conditions are given by the equilibrium of forces and the equality of the corresponding normal displacements. We first employ dual-mixed variational formulations in both domains, which yields the Cauchy stress tensor and the rotation of the solid, together with the gradient of the pressure of the fluid, as the preliminary unknowns. This approach allows us to extend an idea from a recent own work in such a way that both transmission conditions are incorporated now into the definitions of the continuous spaces, and therefore no unknowns on the coupling boundary appear. As a consequence, the pressure of the fluid and the displacement of the solid become explicit unknowns of the coupled problem, and hence two redundant variational terms arising from the constitutive equations, both of them multiplied by stabilization parameters, need to be added for well-posedness. In fact, we show that explicit choices of the above mentioned parameters and a suitable decomposition of the spaces allow the application of the Babuška-Brezzi theory and the Fredholm alternative for concluding the solvability of the resulting augmented formulation. The unknowns of the fluid and the solid are then approximated by a conforming Galerkin scheme defined in terms of Arnold-Falk-Winther and Lagrange finite element subspaces of order 1. The analysis of the discrete method relies on a stable decomposition of the finite element spaces and also on a classical result on projection methods for Fredholm operators of index zero. Finally, numerical results illustrating the theory are also presented..

Key words. mixed finite elements, Arnold-Falk-Winther elements, Helmholtz, elastodynamic.

1. Introduction

In this paper we focus again on the three-dimensional fluid-solid interaction problem studied recently in [20] (see also [17] for the corresponding 2D version). More precisely, our physical model of interest consists of a bounded elastic body (obstacle) Ω_s in $\mathbf{R}^3$ with Lipschitz-continuous boundary Σ, subject to a volume force $\mathbf{F}$, that is fully surrounded by a fluid. Then, given an incident acoustic wave $\mathbf{P}_i$ upon Ω_s, we are interested in determining both the response of the body and the scattered wave. We assume that $\mathbf{P}_i$ and $\mathbf{F}$ exhibit a time-harmonic behaviour with frequency ω and amplitudes p_i and $\mathbf{f}$, respectively, so that p_i satisfies the Helmholtz equation in $\mathbf{R}^3\backslash\Omega_s$. Hence, we may consider that this interaction problem is posed in the frequency domain. In addition, in what follows we let $\boldsymbol{\sigma}_s : \Omega_s \to \mathbf{C}^{3\times 3}$, $\mathbf{u} : \Omega_s \to \mathbf{C}^3$, and $p : \mathbf{R}^3\backslash\Omega_s \to \mathbf{C}$ be the amplitudes of the Cauchy stress tensor,

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the displacement field, and the total (incident + scattered) pressure, respectively, where $\mathbf{C}$ stands for the set of complex numbers.

The fluid is assumed to be perfect, compressible, and homogeneous, with density ρ_f and wave number $\kappa_f := \dfrac{\omega}{v_0}$, where v_0 is the speed of sound in the linearized fluid, whereas the solid is supposed to be isotropic and linearly elastic with density ρ_s and Lamé constants μ and λ. The latter means, in particular, that the corresponding constitutive equation is given by Hooke's law, that is

$$\boldsymbol{\sigma}_s = \lambda \operatorname{tr} \boldsymbol{\varepsilon}(\mathbf{u}) \mathbf{I} + 2 \mu \boldsymbol{\varepsilon}(\mathbf{u}) \qquad \text{in} \quad \Omega_s , \tag{1}$$

where $\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathtt{t}})$ is the strain tensor of small deformations, ∇ is the gradient tensor, tr denotes the matrix trace, $^{\mathtt{t}}$ stands for the transpose of a matrix, and $\mathbf{I}$ is the identity matrix of $\mathbf{C}^{3\times 3}$. Consequently, under the hypotheses of small oscillations, both in the solid and the fluid, the unknowns $\boldsymbol{\sigma}_s$, $\mathbf{u}$, and p satisfy the elastodynamic and acoustic equations in time-harmonic regime, that is:

$$\begin{aligned} \mathbf{div}\, \boldsymbol{\sigma}_s + \kappa_s^2 \mathbf{u} &= -\mathbf{f} \qquad \text{in} \quad \Omega_s , \\ \Delta p + \kappa_f^2 p &= 0 \qquad \text{in} \quad \mathbf{R}^3 \backslash \Omega_s , \end{aligned}$$

where the wave number κ_s of the solid is defined by $\sqrt{\rho_s}\,\omega$, together with the transmission conditions:

$$\begin{aligned} \boldsymbol{\sigma}_s \boldsymbol{\nu} &= -p \boldsymbol{\nu} \qquad \text{on} \quad \Sigma , \\ \rho_f\, \omega^2 \mathbf{u} \cdot \boldsymbol{\nu} &= \frac{\partial p}{\partial \boldsymbol{\nu}} \qquad \text{on} \quad \Sigma , \end{aligned} \tag{2}$$

and the behaviour at infinity given by

$$p - p_i = O(\mathtt{r}^{-1}) \tag{3}$$

and

$$\frac{\partial (p - p_i)}{\partial \mathtt{r}} - \imath \kappa_f (p - p_i) = o(\mathtt{r}^{-1}) , \tag{4}$$

as $\mathtt{r} := \|\mathbf{x}\| \to +\infty$, uniformly for all directions $\dfrac{\mathbf{x}}{\|\mathbf{x}\|}$. Hereafter, $\mathbf{div}$ stands for the usual divergence operator div acting on each row of the tensor, $\|\mathbf{x}\|$ is the euclidean norm of a vector $\mathbf{x} := (x_1, x_2, x_3)^{\mathtt{t}} \in \mathbf{R}^3$, and $\boldsymbol{\nu}$ denotes the unit outward normal on Σ, that is pointing toward $\mathbf{R}^3 \backslash \Omega_s$. The transmission conditions given in (2) constitute the equilibrium of forces and the equality of the normal displacements of the solid and fluid, whereas the equation (4) is known as the Sommerfeld radiation condition.

In the recent work [20] we introduce and analyze a new finite element method for the above interaction problem. Actually, we initially proceed as in [17] and simplify a bit the original model by assuming that the fluid occupies a bounded annular region Ω_f. Hence, a Robin boundary condition imitating the behavior of the scattered field at infinity is imposed on the exterior boundary of Ω_f, which is located far from the obstacle. Then, we employ a dual-mixed formulation for plane elasticity in the solid, in which the elastodynamic equation is used to eliminate the displacement unknown, and keep the usual primal method in the fluid region. Needless to say, avoiding the locking phenomenon that arises in the nearly incompressible case or obtaining direct finite element approximations of the stresses constitute the main reasons for using the dual-mixed method in the solid. Now, the main novelty of our approach in [20] with respect to [17] is the incorporation of the first equation of (2) into the definition of the product space to which the