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ANALYSIS OF THE DISCONTINUOUS GALERKIN INTERIOR PENALTY METHOD WITH SOLENOIDAL APPROXIMATIONS FOR THE STOKES EQUATIONS

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Abstract. The discontinuous Galerkin Interior Penalty Method with solenoidal approximations proposed in [13] for the incompressible Stokes equations is analyzed. Continuity and coercivity of the bilinear form are proved. A priori error estimates, with optimal convergence rates, are derived. 2D and 3D numerical examples with known analytical solution corroborate the theoretical analysis.

Key words. Discontinuous Galerkin, Stokes equations, incompressible flow, divergence-free, Interior Penalty Method, error bounds.

1. Introduction

Discontinuous Galerkin (DG) methods have become very popular for incompressible flow problems, especially in combination with piecewise solenoidal approximations [2, 4, 5, 7, 12, 13, 14, 15]. In the context of conforming finite elements, solenoidal approximations were derived by Crouzeix and Raviart in [6], allowing one to obtain a formulation involving only velocity. Nevertheless their implementation is non-trivial and they are limited to low-order approximations. While alternative solutions for incompressible flows are, among others, velocity-pressure formulations satisfying the Babuska-Brezzi condition, or *hp*-version FEM, in a DG framework high-order solenoidal approximations can be easily defined. This leads to an important saving in the number of degrees of freedom, with the corresponding reduction in computational cost, see [16].

Cockburn and collaborators [4, 5] were among the first researchers to use solenoidal approximations for incompressible flows in the context of the Local Discontinuous Galerkin (LDG) method, and they also introduced the concept of hybrid pressure. Later, the use of solenoidal approximations and hybrid pressure has been applied to an Interior Penalty Method (IPM), in [13], and to a Compact Discontinuous Galerkin (CDG) method, see [16, 17]. In [13], the velocity approximation space is decomposed in every element into a solenoidal part and an irrotational part. This allows for a splitting of the original weak form in two uncoupled problems. The first one solves for velocity and hybrid pressure, and the second one allows evaluating the pressure in the interior of the elements, as a post-processing of the velocity solution.

LDG, CDG and IPM methods all lead to symmetric and coercive bilinear forms for self-adjoint operators. But IPM and CDG methods have the major advantage, relative to LDG, of being compact formulations, that is, the degrees of freedom of one element are only connected to those of immediate neighbors. In [16], IPM and CDG methods are further compared for the solution of the Navier-Stokes equations.

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Both methods present similar results for the accuracy of the numerical solution, reaching optimal convergence rates for velocity and pressure. The main differences are that CDG is less sensitive to the selection of the penalty parameter, but has the major disadvantage of the implementation and computation of the lifting operators. The liftings, introduced in CDG [17] as well as in LDG [4], also induce an approximate orthogonality property and a loss of consistency, whereas IPM is a consistent formulation with a straight-forward implementation.

This paper performs a complete analysis of the discontinuous Galerkin IPM (DG-IPM) with solenoidal approximations and hybrid pressure, as derived and applied to 2D examples in [13], for the incompressible Stokes equations. Standard properties of continuity and coercivity of the obtained weak form are proved. The error estimate for velocity proved in [13] is recalled, and a new error estimate is derived for the post-processed interior pressure, in the case of pure Dirichlet boundary conditions. Some intermediate results from [4, 12] are used in these demonstrations and derivations. All demonstrations in this paper are proved for any spatial dimension (either triangles in 2D or tetrahedra in 3D), except for the unique solvability of the IPM problem, which is only considered for the 2D case.

The paper is structured as follows. The IPM formulation, with a splitting of the velocity space into solenoidal and irrotational parts, is summarized in Section 2 for the solution of the incompressible Stokes equations. Various properties of the IPM formulation are then presented and proved in Section 3. In particular, standard properties of continuity and coercivity of the bilinear form are proved, the error bound for velocity is recalled and a new error bound for interior pressure is derived. 2D and 3D numerical examples with analytical solutions validate the theoretical analysis in Section 4.

2. Discontinuous Galerkin interior penalty formulation for Stokes

Let $\Omega \subset \mathbb{R}^{\mathtt{n_{sd}}}$ be an open bounded domain with boundary $\partial\Omega$ and $\mathtt{n_{sd}}$ the number of spatial dimensions. Suppose that Ω is partitioned in $\mathtt{n_{el}}$ disjoint subdomains Ω_i, triangles in 2D or tetrahedral elements in 3D, with boundaries $\partial\Omega_i$ that define an internal interface Γ; the following definitions and notation are used

$$\overline{\Omega} = \bigcup_{i=1}^{\mathtt{n_{el}}} \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j,$$
$$\text{and } \Gamma := \bigcup_{\substack{i,j=1\\ i\neq j}}^{\mathtt{n_{el}}} \overline{\Omega}_i \cap \overline{\Omega}_j = \Big[\bigcup_{i=1}^{\mathtt{n_{el}}} \partial\Omega_i\Big] \backslash \partial\Omega.$$

The strong form for the steady incompressible Stokes problem can be written as

$$-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = \boldsymbol{s} \quad \text{in } \Omega, \tag{1a}$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1b}$$
$$\boldsymbol{u} = \boldsymbol{u}_D \quad \text{on } \Gamma_D, \tag{1c}$$

where $\Gamma_D = \partial\Omega$, $\boldsymbol{s} \in \mathcal{L}_2(\Omega)$ is a source term, and $\boldsymbol{\sigma}$ is the ("dynamic" or "density-scaled") Cauchy stress, which is related to velocity $\boldsymbol{u}$, and pressure p, by the linear Stokes' law

$$\boldsymbol{\sigma} = -p\,\mathbf{I} + 2\nu\,\boldsymbol{\nabla}^{\mathbf{S}}\boldsymbol{u}, \tag{2}$$

with ν being the kinematic viscosity and $\boldsymbol{\nabla}^{\mathbf{S}} = \frac{1}{2}(\boldsymbol{\nabla} + \boldsymbol{\nabla}^T)$.