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ERROR ANALYSIS OF A MIXED FINITE ELEMENT METHOD FOR THE MONGE-AMPÈRE EQUATION

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Abstract. We analyze the convergence of a mixed finite element method for the elliptic Monge-Ampère equation in dimensions 2 and 3. The unknowns in the formulation, the scalar variable and a discrete Hessian, are approximated by Lagrange finite element spaces. The method originally proposed by Lakkis and Pryer can be viewed as the formal limit of a Hermann-Miyoshi mixed method proposed by Feng and Neilan in the context of the vanishing moment methodology. Error estimates are derived under the assumption that the continuous problem has a smooth solution.

Key words. Monge-Ampère, mixed finite elements, Lagrange elements, fixed point.

1. Introduction

We are interested in the numerical approximation of convex solutions of the nonlinear elliptic Monge-Ampère equation

$$\begin{aligned} \det D^2 u &= f \text{ in } \Omega \\ u &= g \text{ on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here Ω is a convex polygonal domain of $\mathbb{R}^d$ and $f \in C(\Omega), g \in C(\partial\Omega)$ with $f \geq c_0 > 0$ for a constant $c_0 > 0$. We give an analysis of a mixed finite element approximation of (1.1) for dimensions $d = 2$ and $d = 3$. The unknowns in the formulation are the scalar variable and a discrete Hessian and both are approximated by Lagrange finite element spaces of degree $k \geq 1$.

The numerical study of Monge-Ampère type equations is a recent active research area where it appears that techniques to prove convergence to the so-called viscosity solutions of (1.1) are inherently different from the ones needed to derive error estimates for smooth solutions. It has been documented in [7, 8] for the two-dimensional problem that the method of Lakkis and Pryer with Lagrange elements of degree $k \geq 2$ captures viscosity solutions of the Monge-Ampère equation. Some numerical methods proposed for the Monge-Ampère equation, e.g. [3], do not perform well for non smooth solutions when the discrete problem is solved by Newton's method. On the other hand, with the mixed method one can use Newton's method and still have numerical convergence for non smooth solutions. This offers the possibility of numerical solvers faster than the iterative methods proposed in [1]. In this paper we assume that (1.1) has a smooth solution.

To guarantee the existence of a smooth solution, one has to assume that the domain is smooth and strictly convex and the data f and g are also smooth [9]. The convex polygonal domain may be assumed to be an approximation of a smooth and strictly convex domain. Another approach would be to consider elements with curved faces and enforce Dirichlet boundary conditions by a penalty method as in [3].

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The method of Lakkis and Pryer has been recently generalized in [8] where a discontinuous finite element space is used to approximate the discrete Hessian. This results in a more efficient numerical method and an analysis of both types of methods were given in [8] for the two dimensional problem. The connection of the method of Lakkis and Pryer with a Herman-Miyoshi mixed finite element was also noted in [8]. But the idea to analyze the method from the point of view of mixed methods, or to view it as the formal limit of the mixed method proposed in the context of the vanishing moment methodology in [6], was not considered. One possible reason is that Herman-Miyoshi type mixed methods were originally studied for equations involving the biharmonic operator. Several technical arguments have to be made as the linearized Monge-Ampère equation is a second order elliptic equation. The contributions of this paper are:

(1) An analysis valid in both dimensions 2 and 3 and different from the one given in [8] for the two dimensional problem.
(2) Error estimates for Lagrange elements of degree $k \geq 3$ in dimensions 2 and 3.
(3) Numerical experiments for smooth solutions and Lagrange elements of degree $k = 1$. Previous authors in their implementation eliminated the discrete Hessian, which does not necessarily converge for $k = 1$, and concluded the divergence of the method for linear elements.

The approach taken in this paper could help in the investigation of the method for low order elements, i.e. for $k = 1, 2$.

The paper is organized as follows. In the second section we introduce some notations, recall classical finite element results, present the mixed method for the Monge-Ampère equation and useful facts about computations with determinants. Our variational formulation is well posed for dimensions $d = 2$ and $d = 3$ but other general statements are valid for arbitrary dimension d. In section 3 we give the error analysis. The last section is devoted to the numerical results.

2. Preliminaries

2.1. Notation and assumptions. Let Ω be an open convex bounded subset of $\mathbb{R}^d$ with boundary $\partial\Omega$ and let $\mathcal{T}_h$ denote a triangulation of Ω into simplices K. We denote by h_K the diameter of the element K and $h = \max_{K\in\mathcal{T}_h} h_K$. We make the assumption that the triangulation is conforming and satisfies the usual shape regularity condition, i.e. there exists a constant $\sigma > 0$ such that $h_K/\rho_K \leq \sigma$, for all $K \in \mathcal{T}_h$ where ρ_K denotes the radius of the largest ball inside K. To be able to use global inverse estimates, c.f. (2.2) and (2.3) below, we require the triangulation to be also quasi-uniform, i.e. there is a constant $C > 0$ such that $h \leq Ch_K$ for all $K \in \mathcal{T}_h$.

We use the usual notation $L^p(\Omega), 2 \leq p \leq \infty$ for the Lebesgue spaces and $H^s(\Omega), 1 \leq s < \infty$ for the Sobolev spaces of elements of $L^2(\Omega)$ with weak derivatives of order less than or equal to s in $L^2(\Omega)$. We recall that $W^{s,\infty}(\Omega)$ is the Sobolev space of functions with weak derivatives of order less than or equal to s in $L^\infty(\Omega)$. For a given normed space X, we denote by X^d the space of vector fields with components in X and by $X^{d\times d}$ the space of matrix fields with each component in X. The norm in X is denoted by $|| \, ||_X$ and we omit the subscripts Ω, d, and $d \times d$ when it is clear from the context. We will use the standard notation $| \, |_{H^s}$ for the semi norm on $H^s(\Omega), H^s(\Omega)^d$ and $H^s(\Omega)^{d\times d}$. The inner product in $L^2(\Omega), L^2(\Omega)^d$, and $L^2(\Omega)^{d\times d}$ is denoted by $(,)$ and we use $\langle , \rangle$ for the inner product on $L^2(\partial\Omega)$