INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 11, Number 4, Pages 787–815

POLLUTION-FREE FINITE DIFFERENCE SCHEMES FOR NON-HOMOGENEOUS HELMHOLTZ EQUATION

KUN WANG AND YAU SHU WONG

Abstract. In this paper, we develop pollution-free finite difference schemes for solving the non-homogeneous Helmholtz equation in one dimension. A family of high-order algorithms is derived by applying the Taylor expansion and imposing the conditions that the resulting finite difference schemes satisfied the original equation and the boundary conditions to certain degrees. The most attractive features of the proposed schemes are: first, the new difference schemes have a $2n-$order of rate of convergence and are pollution-free. Hence, the error is bounded even for the equation at high wave numbers. Secondly, the resulting difference scheme is simple, namely it has the same structure as the standard three-point central differencing regardless the order of accuracy. Convergence analysis is presented, and numerical simulations are reported for the non-homogeneous Helmholtz equation with both constant and varying wave numbers. The computational results clearly confirm the superior performance of the proposed schemes.

Key words. Helmholtz equation, Finite difference method, Convergence analysis, High wave number, Pollution-free, High-order schemes.

1. Introduction

In the study of time-harmonic wave propagations in one dimension, if we assume the wave has a steady-state and its circular frequency is fixed, we obtain the Helmholtz equation. The model equation to be investigated in this paper is given by:

$$-u_{xx}(x) - k^2u(x) = f(x),\ \ x \in (0, 1), \tag{1}$$

$$u(0) = 0, \tag{2}$$

$$u_x(1) - iku(1) = 0, \tag{3}$$

where $k = \omega/c$ is the wave number with ω being the circular frequency, c and f represents the speed of sound and the forcing term, respectively.

The Helmholtz equation arises in many problems related to wave propagations, such as acoustic, electromagnetic wave scattering and geophysical applications. It has been accepted that it is a difficult computational problem to develop efficient and accurate numerical schemes to solve the Helmholtz equation at high wave numbers.

The foremost difficulty in the numerical solution for the Helmholtz equation is to eliminate or minimize the "pollution effect" which causes a serious problem as the wave number k increases [10, 17, 25]. When the wave number is small, there is no difficulty to obtain accurate numerical solution for the Helmholtz equation. However, the accuracy of the computed solution deteriorates rapidly for problems at high wave numbers. Thus, eliminating or improving the pollution term will be a

Received by the editors February 9, 2014.

2000 *Mathematics Subject Classification.* 65N06, 65N15, 65N22.

This work was supported by the Pacific Institute for the Mathematical Sciences, the Natural Sciences and Engineering Research Council of Canada and the Natural Science Foundation of China (No. 11201506), the Fundamental Research Funds for the Central Universities (Project No. 106112013001).

crucial issue in developing efficient and accurate numerical schemes for (1)-(3). To overcome this difficulty, many literatures have been reported in the past decades, and the reader is referred to [21, 22, 29, 34, 35, 41, 45, 5, 6, 16, 4] for the finite difference method and [3, 1, 2, 10, 15, 17, 20, 23, 24, 25, 26, 27, 28, 37, 39] for the finite element approximation. In fact, the following two issues are critical to the "pollution effect". First, when approximating the Helmholtz equation numerically, the "numerical wave number" $\tilde{k}$ from a resulting computational scheme will disperse in the non-dispersive media and they may not be the same as the wave number k from the original equation, which results the numerical dispersion. It is well-known that for the standard finite difference method, the difference

$$|\tilde{k} - k| \leq C_1 k^3 h^2, \tag{4}$$

and for the fourth order compact scheme proposed in [21]

$$|\tilde{k} - k| \leq C_1 k^5 h^4, \tag{5}$$

where C_1 is a general constant independent of $k, \tilde{k}$ and the mesh size h. Hereafter, we use C_1 to denote a general constant independent of $k, \tilde{k}$ and the mesh size h, but it may take different values at its different occurrences. For the p-version finite element method (see [25]), we have

$$|\tilde{k} - k| \leq C_1 k \Big(\frac{hk}{2p}\Big)^{2p}. \tag{6}$$

Ainsworth [2] improved the estimate for the p-version finite element method result:

$$|\tilde{k} - k| \leq C_1 k \Big(\frac{p!}{(2p)!}\Big)^2 \Big(\frac{hk}{2p+1}\Big)^{2p}. \tag{7}$$

Recently, Zhu et al. [37] reported a better estimate by using the continuous interior penalty finite element method

$$|\tilde{k} - k| \leq C_1 kh. \tag{8}$$

The above results reveal the relationship between the original wave number k and the "numerical wave number" $\tilde{k}$. For a fixed kh, we observe that the difference increases as k increases for all methods.

Another important consequence is that the "pollution effect" has a direct impact on the error estimate. In the finite difference approach, Singer and Turkel [35] proposed a fourth order scheme based on the Pade approximation. Higher order difference schemes had also been investigated in [22, 29, 34, 35, 41]. However, the error estimates have not been analyzed. Recently, Fu presented the error estimate for a compact fourth order finite difference method in [21]. Although it has been claimed that the developed compact scheme is independent of the wave number, numerical simulations reveal that all finite difference methods referred above depend on the wave number and the "pollution effect" will become more serious as k increases. For the finite element method, stabilities and error estimates are analyzed for the problem (1)-(3) with PML boundary in [23]. In [10], Babuška et al. developed a generalized finite element method such that the pollution effect is minimal. Let h be the step size and assume that kh is fixed, Ihlenburg and Babuška proposed the $h-p$ version finite element method in [25, 27, 28], the error estimates and dispersion analysis confirm that the "pollution effect" can be reduced as p increases or h decreases. This estimate is further improved in [15]. Recently, the continuous interior penalty finite element method has been proposed in [37]. Although a modified finite element method is considered in [39], a physical spline finite element method is investigated in [20] and a least squares finite element method with high