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A PARALLEL VARIATIONAL MULTISCALE METHOD FOR INCOMPRESSIBLE FLOWS BASED ON THE PARTITION OF UNITY

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Abstract. A parallel variational multiscale method based on the partition of unity is proposed for incompressible flows in this paper. Based on two-grid method, this algorithm localizes the global residual problem of variational multiscale method into a series of local linearized residual problems. To decrease the undesirable effect of the artificial homogeneous Dirichlet boundary condition of local sub-problems, an oversampling technique is also introduced. The globally continuous finite element solutions are constructed by assembling all local solutions together using the partition of unity functions. Numerical simulations demonstrate the high efficiency and flexility of the new algorithm.

Key words. Incompressible flows, variational multiscale method, local and parallel, partition of unity, oversampling.

1. Introduction

The variational multiscale method was proposed to solve multiscale problems by Hughes and co-workers in [1, 2]. A projection of the large scales in Large Eddy Simulation method into appropriate subspaces was introduced. Since then much attention has been paid in this field. For example, John and Kaya [3] gave the finite element analysis of a variational multiscale method for the Navier-Stokes equations. Gravemeier et al. [4] also presented the three-level variational multiscale method. Zheng et al. improved the finite element variational multiscale method by introducing two Gauss integration method [5] and adaptive technique [6]. Zhang et al. [7], Yu et al. [8], Shan et al. [9] et al. presented subgrid model, projection basis and modular type to improve the variational multiscale methods, respectively.

Based on the observation that in numerical simulations low frequency components can be approximated well by the relative coarse grid and high frequency components can be computed on a fine grid by some local and parallel procedure, the parallel finite element computations have been widely used [10, 11, 12, 13]. Combining the partition of unity method [14, 15] and the parallel adaptive algorithm from [11], Holst [16, 17] constructed the parallel partition of unity method (PPUM). Zheng et al. [19, 20] developed some local and parallel finite element algorithms based on the partition of unity. Song et al. [18] presented an adaptive local postprocessing technique based on the partition of unity method for the Navier-Stokes equations. There are also some papers improving the variational multiscale methods by combining with two-grid method or local and parallel techniques [21, 22].

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It is natural to consider to add the local parallel method to the variational multiscale method in order to retain the best features of both methods and overcome many of their defects. In particular, we use the variational multiscle method based on two local Gauss integrations [5] since it avoids constructing the projection operator, keeps the same efficiency and does not need extra storage compared with common VMS method. Comparing with the parallel method in [22], we add an artificial stabilization term in the local and parallel procedure by considering the residual as a subgrid value, which keeps the sub-problems stable. Then, an oversampling technique is introduced in order to overcome the undesirable effect of the artificial homogeneous Dirichlet boundary conditions of local sub-problems. The interesting points in this algorithm lie in: firstly, a class of partition of unity is derived by a given triangulation, which guides the domain decomposition; secondly, the series of local linearized residual problems are implemented in parallel, and they require less communication between each other; finally, the globally continuous finite element solution is obtained by assembling all local solutions together via the partition of unity functions.

The outline of the paper is as follows. We introduce the Navier-Stokes equations, the notations and some well-known results for the finite element methods in section 2. In section 3, we first propose the parallel variational multiscale method based on the partition of unity and then derive the error estimates. In section 4, the implementation and some numerical simulations are presented to illustrate the efficiency of our method. And finally a short conclusion is presented in section 5.

2. The Navier-Stokes Equations

We consider the following incompressible flows

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \;\; \texttt{in}\ \Omega, \\ \nabla\cdot\mathbf{u} &= 0 \;\; \texttt{in}\ \Omega, \\ \mathbf{u} &= 0 \;\; \texttt{on}\ \partial\Omega, \end{aligned} \tag{1}$$

where Ω represents a polyhedral domain in R^d (d =2, 3) with boundary $\partial\Omega$, $\mathbf{u}$, p, $\mathbf{f}$ and $\nu > 0$ represent the velocity vector, pressure, prescribed body force, kinematic viscosity respectively. And ν is inversely proportional to the Reynolds number Re.

For a bounded domain $\Omega \subset R^d$, we use the standard notations for Sobolev spaces $W^{s,k}(\Omega)$ and their associated norms [23, 24]. Especially when $k = 2$, $H^s(\Omega) = W^{s,2}(\Omega)$ denotes the usual Soblev space, $\|\cdot\|_{s,\Omega} = \|\cdot\|_{s,2,\Omega}$ denotes standard Soblev norm, $(\cdot,\cdot)_s$ denotes the inner product in $L^2(\Omega)$ or its vector value version. The space $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$ is equipped with the usual norm $\|\nabla\cdot\|_{0,\Omega}$ or its equivalent norm $\|\cdot\|_{1,\Omega}$ due to the Poincare's inequality. $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$. In the following we will denote the spaces consisting of vector-valued functions in boldface.

For sub-domains $D \subset G \subset \Omega$, $D \subset\subset G$ means that $\text{dist}(\partial D\backslash\partial\Omega, \partial G\backslash\partial\Omega) > 0$. Throughout the paper we use C to denote a generic positive constant whose value may change from place to place but remains independent of the mesh parameter h.

The standard variational formulation of (1) is given by: find $(\mathbf{u}, p) \in (\mathbf{X}, M)$ satisfying

$$\nu a(\mathbf{u},\mathbf{v}) + b(\mathbf{u},\mathbf{u},\mathbf{v}) - d(\mathbf{v},p) + d(\mathbf{u},q) = (\mathbf{f},\mathbf{v}), \quad \forall(\mathbf{v},q) \in (\mathbf{X}, M), \tag{2}$$

where

$$\mathbf{X} = \mathbf{H}_0^1(\Omega), \quad M = L_0^2(\Omega) = \{q \in L^2(\Omega); \int_\Omega q\mathbf{dx} = 0\},$$