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WEAK GALERKIN FINITE ELEMENT METHODS ON POLYTOPAL MESHES

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Abstract. This paper introduces a new weak Galerkin (WG) finite element method for second order elliptic equations on polytopal meshes. This method, called WG-FEM, is designed by using a discrete weak gradient operator applied to discontinuous piecewise polynomials on finite element partitions of arbitrary polytopes with certain shape regularity. The paper explains how the numerical schemes are designed and why they provide reliable numerical approximations for the underlying partial differential equations. In particular, optimal order error estimates are established for the corresponding WG-FEM approximations in both a discrete H^1 norm and the standard L^2 norm. Numerical results are presented to demonstrate the robustness, reliability, and accuracy of the WG-FEM. All the results are established for finite element partitions with polytopes that are shape regular.

Key words. weak Galerkin, finite element methods, discrete gradient, second-order elliptic problems, polytopal meshes

1. Introduction

In this paper, we are concerned with a further and new development of weak Galerkin (WG) finite element methods for partial differential equations. Our model problem is a second-order elliptic equation which seeks unknown function $u = u(x)$ satisfying

$$-\nabla \cdot (a(x, u, \nabla u)\nabla u) = f(x), \quad \text{in } \Omega, \tag{1}$$

where Ω is a polytopal domain in $\mathbb{R}^d$ (polygonal or polyhedral domain for $d = 2, 3$), ∇u denotes the gradient of the function $u = u(x)$, and $a = a(x, u, \nabla u)$ is a symmetric $d \times d$ matrix-valued function in Ω. We shall assume that the differential operator is strictly elliptic in Ω; that is, there exists a positive number $\lambda > 0$ such that

$$\xi^t a(x, \eta, p)\xi \geq \lambda \xi^t \xi, \qquad \forall \xi \in \mathbb{R}^d, \tag{2}$$

for all $x \in \Omega, \eta \in \mathbb{R}, p \in \mathbb{R}^d$. Here ξ is understood as a column vector and ξ^t is the transpose of ξ. We also assume that the differential operator has bounded coefficients; that is for some constant Λ we have

$$|a(x, \eta, p)| \leq \Lambda, \tag{3}$$

for all $x \in \Omega, \eta \in \mathbb{R}$, and $p \in \mathbb{R}^d$.

Introduce the following form

$$\mathfrak{a}(\phi; u, v) := \int_\Omega a(x, \phi, \nabla \phi)\nabla u \cdot \nabla v dx. \tag{4}$$

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For simplicity, let the function f in (1) be locally integrable in Ω. We shall consider solutions of (1) with a non-homogeneous Dirichlet boundary condition

$$u = g, \quad \text{on } \partial\Omega, \tag{5}$$

where $g \in H^{\frac{1}{2}}(\partial\Omega)$ is a function defined on the boundary of Ω. Here $H^1(\Omega)$ is the Sobolev space consisting of functions which, together with their gradients, are square integrable over Ω. $H^{\frac{1}{2}}(\partial\Omega)$ is the trace of $H^1(\Omega)$ on the boundary of Ω. The corresponding weak form seeks $u \in H^1(\Omega)$ such that $u = g$ on $\partial\Omega$ and

$$\mathfrak{a}(u; u, v) = F(v), \qquad \forall v \in H_0^1(\Omega), \tag{6}$$

where $F(v) \equiv \int_\Omega f v dx$.

Galerkin finite element methods for (6) refer to numerical techniques that seek approximate solutions from a finite dimensional space V_h consisting of piecewise polynomials on a prescribed finite element partition $\mathcal{T}_h$. The method is called conforming if V_h is a subspace of $H^1(\Omega)$. Conforming finite element methods are then formulated by solving $u_h \in V_h$ such that $u_h = I_h g$ on $\partial\Omega$ and

$$\mathfrak{a}(u_h; u_h, v) = F(v), \qquad \forall v \in V_h \cap H_0^1(\Omega), \tag{7}$$

where $I_h g$ is a certain approximation of the Dirichlet boundary value. When V_h is not a subspace of $H^1(\Omega)$, the form $\mathfrak{a}(\phi; u, v)$ is no longer meaningful since the gradient operator is not well-defined for non-H^1 functions in the classical sense. Nonconforming finite element methods arrive when the gradients in $\mathfrak{a}(\phi; u, v)$ are taken locally on each element where the finite element functions are polynomials. More precisely, the form $\mathfrak{a}(\phi; u, v)$ in nonconforming finite element methods is given element-by-element as follows

$$\mathfrak{a}_h(\phi; u, v) := \sum_{T \in \mathcal{T}_h} \int_T a(x, \phi, \nabla\phi) \nabla u \cdot \nabla v dx. \tag{8}$$

When V_h is close to be conforming, the form $\mathfrak{a}_h(\phi; u, v)$ shall be an acceptable approximation to the original form $\mathfrak{a}(\phi; u, v)$. The key in the nonconforming method is to explore the maximum non-conformity of V_h when the approximate form $\mathfrak{a}_h(\phi; u, v)$ is required to be sufficiently close to the original form.

A natural generalization of the nonconforming finite element method would occur when the following extended form of (8) is employed

$$\mathfrak{a}_w(\phi; u, v) := \sum_{T \in \mathcal{T}_h} \int_T a(x, \phi, \nabla_w \phi) \nabla_w u \cdot \nabla_w v dx, \tag{9}$$

where ∇_w is an approximation of ∇ locally on each element. By viewing ∇_w as a weakly defined gradient operator, the form $\mathfrak{a}_w(\phi; u, v)$ would give a new class of numerical methods called *weak Galerkin (WG)* finite element methods.

In general, weak Galerkin refers to finite element techniques for partial differential equations in which differential operators (e.g., gradient, divergence, curl, Laplacian) are approximated by weak forms as distributions. In [20], a WG method was introduced and analyzed for second order elliptic equations based on a *discrete weak gradient* arising from local *RT* [18] or *BDM* [9] elements. Due to the use of the *RT* and *BDM* elements, the WG finite element formulation of [20] was limited to classical finite element partitions of triangles ($d = 2$) or tetrahedra ($d = 3$). In [21], a weak Galerkin finite element method was developed for the second order elliptic equation in the mixed form. The use of stabilization for the flux variable in the mixed formulation is the key to the WG mixed finite element method of [21]. The resulting WG mixed finite element schemes turned out to be applicable for general