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NUMERICAL APPROXIMATION OF VISCOELASTIC FLOWS IN AN ELASTIC MEDIUM

HYESUK LEE AND SHUHAN XU

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Abstract. We study the numerical approximation of a flow problem governed by a viscoelastic model coupled with a one-dimensional elastic structure equation. A variational formulation for flow equations is developed based on the Arbitrary Lagrangian-Eulerian (ALE) method and stability of the time discretized system is investigated. For decoupled numerical algorithms we consider both an explicit scheme of the Leap-Frog type and a fully implicit scheme.

Key words. viscoelastic flow, elastic structure, finite elements.

1. Introduction

There have been extensive numerical studies on multidisciplinary problems where a coupled mathematical model is present for more than one media. In particular, the interaction of fluid flows with an elastic medium is of great interest for both industrial and biological uses. Examples of such systems include bronchial air ways [10], blood flow in arteries [4, 21] and micro-fluidic devices [25]. A fluid-elastic system is typically highly linked through a strong coupling of governing equations, making its mathematical and numerical study very challenging. Subsystems interact with each other in the manner that the stress of the fluid imposes forces on the boundaries of the elastic medium, which causes continuous displacement of the flexible boundaries, resulting in a movement of the fluid domain. Such an interaction problem is mathematically described as a system of time-dependent, coupled, linear/nonlinear partial differential equations with moving boundaries.

Modeling and simulation of the fluid-structure interaction have been studied by many researchers [3, 5, 7, 27]. Fluid-structure interactions involving elementary fluids, governed by equations for a potential functions, e.g., the Laplace equation or wave equation, were studied in [3, 11]. Studies on multi-phase systems involving viscous, incompressible fluids and elastic solids can be found in [5, 17, 21]. One important application of such a system is a blood flow. Interactions of blood flow with a vessel wall are modeled by coupled fluid-structure equations with matching conditions on velocity and traction along the interface. The vessel wall is known to be elastic material, thus a linear/nonlinear elastic equation has been used for the structural mechanics. The blood flow has been modeled by the Stokes or the Navier- Stokes equations in literature [4, 13, 21, 22], although blood is known to be non-Newtonian in general. In particular, it is well known that blood flow in small vessels and capillaries behaves as a viscoelastic fluid [19, 28].

There are only a few numerical studies found in literature for the non-Newtonian fluids coupled with elastic solids. Some simulation results for the interaction of non-Newtonian fluids with deformable bodies were reported in engineering journals [1, 26] for the purpose of model validation. Even though there are some reports [15,

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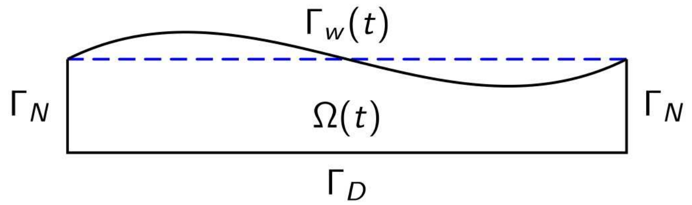

FIGURE 1. Fluid domain.

18] on numerical methods for simulation of blood flows using non-Newtonian fluid models, detailed numerical analysis such as studies on stability of time-stepping schemes is, in general, lacking from the current literature.

Mathematical study for partial differential equations governing a viscoelastic fluid behavior is still far behind compared to advances in computing. It is well-known that an analytical or numerical study of viscoelastic flows is very challenging due to complexity of governing equations. One of the difficulties in simulating viscoelastic flows arises from the hyperbolic nature of the constitutive equation for which one needs to use a stabilization technique such as the streamline upwinding Petrov-Galerkin (SUPG) [24] method or the discontinuous Galerkin method [2].

In [6, 16] we introduced a modified Johnson-Segalman model, referred to as the Oseen-viscoelastic model, in which the velocity in the nonlinear pieces of the constitutive equation was taken to be a known function. For this model we have been able to provide a rigorous mathematical analysis of the model equations and their approximation, and provide some additional insights into the investigated problems (high Weissenberg number problem, domain decomposition, optimal control) in viscoelasticity. We use the Oseen model for analysis in this paper, however numerical tests will be for the standard Jonson-Segalman model. For the structure model, we use a one-dimensional string model introduced in [20, 21].

The remainder of this paper is organized as follows. In the next section model equations are described and stability of the coupled system is proved. The variational formulation of the problem based on the *Arbitrary Lagrangian Eulerian* is introduced in Section 3. In Section 4 the time discretized variational formulation and its stability are studied, and finally numerical results are presented in Section 5.

2. Model equations

We consider a viscoelastic flow problem, where flow equations are coupled with a one-dimensional elastic structure model. Let $\Omega(t)$ be a bounded domain at time t in $\mathbb{R}^2$ with the Lipschitz continuous boundary $\Gamma(t)$. Suppose $\Gamma(t)$ consists of three parts: Γ_D, Γ_N and $\Gamma_w(t)$, where $\Gamma_D \cup \Gamma_N$ is a fixed boundary and $\Gamma_w(t)$ a moving wall boundary.

Consider the viscoelastic model equations:

$$\boldsymbol{\sigma} + \lambda \left(\frac{\partial \boldsymbol{\sigma}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\sigma} + g_\beta(\boldsymbol{\sigma}, \nabla \mathbf{u}) \right) - 2\,\alpha\, D(\mathbf{u}) = \mathbf{0} \quad \text{in } \Omega(t)\,, \tag{1}$$

$$Re \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) - \nabla \cdot \boldsymbol{\sigma} - 2(1-\alpha)\, \nabla \cdot D(\mathbf{u}) + \nabla p = \mathbf{f} \quad \text{in } \Omega(t)\,, \tag{2}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega(t)\,, \tag{3}$$